Calculus

December 15, 2010

1. (12 points) Let $f(x) = x^2 - 2x$. (a) Compute $f(x+h)$. (b) Form $\frac{f(x+h)-f(x)}{h}$ (c) Compute $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and state its relation to $f'(x)$
2. (12 points) Find the derivatives of the following functions
   (a) (a) $f(x) = (5x^2 - 1)(x^3 + 3)$. (b) $f(x) = \frac{x}{3x-4}$. (c) $f(x) = \sqrt{x^2 + x + 1}$
3. (8 points) Find the second derivatives of the following functions
   (a) $f(x) = x^5 - x^4 + 4x^3 - 2x^2 + x - 8$
   (b) $f(x) = x^{2/3}$
4. (8 points) Find the slope and an equation of the tangent line to the graph $f(x) = 2x + 1/\sqrt{x}$ at the point $(1, 3)$
5. (24 points) Suppose the relationship between the unit price p on dollars and the quality demanded of the Acrosonic model F loudspeaker system is given

   $$p(x) = -0.02x + 400,\ 0 \le x \le 20,000,$$

   and the cost of producing x units is

   $$C(x) = 100x + 200,000$$

   (a) Find the revenue function R
   (b) Find the marginal revenue function R'
   (c) Compute $R'(2000)$ and interpret your result
   (d) Find the profit function P.
   (e) Find the marginal profit function P'
   (f) Compute $P'(2000)$ and interpret your result.
6. (12 points) The function

   $$I(x) = -0.3t^3 + 3t^2 + 100, (0 \le t \le 9)$$

   gives the CPI (consumer price index) of an economy

   (a) Find the inflation rate at $t = 6$
   (b) Show that inflation is moderating at $t = 6$
   (c) Find the point of inflection of $I(x)$ and discuss its significance
7. (8 points) Approximate $\sqrt{26.5}$ using differentials
8. (12 points) $f(x) = x^3 - 3x^2 - 24x + 32$
   (a) Determine the intervals where $f(x)$ is increasing and where it is decreasing
   (b) Find the relative maxima and relative minima of $f(x)$
   (c) Determine the intervals where $f(x)$ is concave upward and where it is concave downward
9. (10 points) By cutting away identical squares from each corner of a rectangular piece of cardboard and folding up the resulting flaps, the cardboard may be turned into an open box. If the cardboard is 16 inches long and 10 inches wide, find the dimensions of the box that will yield the maximum volume.

#1. $f(x) = x^2 - 2x$.

(a) $f(x+h) = (x+h)^2 - 2(x+h) = x^2+2xh+h^2-2x-2h$

(b) $\frac{f(x+h)-f(x)}{h} = \frac{\cancel{x^2}+2xh+h^2-\cancel{2x}-2h-\cancel{x^2}+\cancel{2x}}{h} = \frac{\cancel{h}(2x+h-2)}{\cancel{h}} = 2x+h-2$

(c) $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 2x+h-2 = 2x-2$.

$f'(x) = 2x-2$.

$\therefore \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$

#2. (a) $f(x) = (5x^2-1)(x^3+3)$

$f'(x) = 10x(x^3+3) + (5x^2-1)(3x^2) = x(10x^3+30+15x^3-3x) = x(25x^3-3x+30)$.

(b) $f(x) = \frac{x}{3x-4}$

$f'(x) = \frac{3x-4-3x}{(3x-4)^2} = \frac{-4}{(3x-4)^2}$

(c) $f(x) = \sqrt{x^2+x+1} = (x^2+x+1)^{1/2}$.

$f'(x) = \frac{1}{2}(x^2+x+1)^{-\frac{1}{2}}(2x+1) = \frac{2x+1}{2\sqrt{x^2+x+1}}$

#3

(a) $f(x) = x^5 - x^4 + 4x^3 - 2x^2 + x - 8$

$f'(x) = 5x^4 - 4x^3 + 12x^2 - 4x + 1$

$f''(x) = 20x^3 - 12x^2 + 24x - 4$. or $= 4(5x^3-3x^2+6x-1)$

(b) $f(x) = x^{\frac{2}{3}}$.

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$, $\quad f''(x) = -\frac{2}{9}x^{-\frac{4}{3}} = -\frac{2}{9x^{4/3}}$

#4 $f(x) = 2x + \frac{1}{\sqrt{x}} = 2x + x^{-\frac{1}{2}}$

$f'(x) = 2 + (-\frac{1}{2})x^{-\frac{3}{2}}$, $\quad f'(1) = 2 - \frac{1}{2} = \frac{3}{2}$ is the slope

tangent line : $y - 3 = \frac{3}{2}(x-1) \Rightarrow y = \frac{3}{2}x + \frac{3}{2}$

#5 P.203.

$p = -0.02x + 400$, $0 \le x \le 20{,}000$.

$C(x) = 100x + 200{,}000$.

(a) revenue function $R(x) = px$.

$R(x) = (-0.02x + 400)x = -0.02x^2 + 400x$. $(0 \le x \le 20{,}000)$

(b) $R'(x) = -0.04x + 400$.

(c) $R'(2000) = -80 + 400 = 320$,

the actual to be revenue to be realized from the sale of the 2001st is approximately \$320

(d) profit function $P(x) = R(x) - C(x)$

$= -0.02x^2 + 400x - 100x - 200{,}000$

$= -0.02x^2 + 300x - 200{,}000$

(e) $P'(x) = -0.04x + 300$.

(f) $P'(2000) = -80 + 300 = 220$.

the actual profit realized from the sale of the 2001st is approximately \$220.

#6. P.215 and P.273.

$I(t) = -0.3t^3 + 3t^2 + 100$, $(0 \le t \le 9)$

(a) $I'(t) = -0.9t^2 + 6t$

$I'(6) = -0.9(36) + 36 = -32.4 + 36 = 3.6$.

$I(6) = -0.3(216) + 3(36) + 100 = -64.8 + 108 + 100 = 143.2$.

The inflation rate is $\frac{I'(6)}{I(6)} = \frac{3.6}{143.2} \doteqdot 0.0251 \Rightarrow 2.5\%$.

(b) $I''(t) = -1.8t + 6$.

$I''(6) = -10.8 + 6 = -4.8$. $\therefore I''(6) < 0$.

Hence inflation is moderating at $t=6$.

(c) $I''(t) = -1.8t + 6 = 0$, $t = \frac{60}{18} = \frac{10}{3}$.

$I(\frac{10}{3}) = -0.3(\frac{10}{3})^3 + 3(\frac{10}{3})^2 + 100 = \cancel{-0.3}^{-0.1} \cdot \frac{1000}{\cancel{27}_{9}} + 3 \cdot \frac{100}{\cancel{9}_{3}} + 100 = \frac{-100 + 300 + 900}{9} = \frac{1100}{9}$.

the point of inflection of $I(x)$ is $(\frac{10}{3}, \frac{1100}{9})$, <解釋看課本 p.273 >.

1. p.235.

Let $y = f(x) = \sqrt{x}$, 25 is nearest 26.5.

$\therefore$ take $x = 25$. $\Delta x = 26.5 - 25 = 1.5$.

$\Delta y \approx dy = f'(x)\,\Delta x = \left(\frac{1}{2\sqrt{x}}\Big|_{x=25}\right) 1.5 = \frac{1}{10} \times 1.5 = 0.15$.

$\sqrt{26.5} - \sqrt{25} = \Delta y \approx 0.15$.

$\Rightarrow \sqrt{26.5} = \sqrt{25} + 0.15 = 5 + 0.15 = 5.15$

3 p.256 , p.249. p.269.

$f(x) = x^3 - 3x^2 - 24x + 32$.

a) $f'(x) = 3x^2 - 6x - 24 = 0$. $\Rightarrow 3(x^2 - 2x - 8) = 3(x-4)(x+2) = 0$.

$x = 4, -2$ is cirticle point.

$+ + + + \; 0 \; - - - \; 0 \; + + +$ $f'(x)$; -2, 4

• $f'(x) < 0$ decreasing

$f'(x) > 0$ increasing

$\therefore$ increasing : $(-\infty, -2)$, $(4, \infty)$

decreasing : $(-2, 4)$.

b) $f(4) = 64 - 48 - 96 + 32 = -48$.

$f(-2) = -8 - 12 + 48 + 32 = 60$.

$\therefore$ the relative maximum at $x = -2$ is 60 $(-2, 60)$

the relative minimum at $x = 4$ is -48 , $(4, -48)$

c) $f''(x) = 6x - 6 = 0 \Rightarrow 6(x-1) = 0$.

when $x = 1$, $f''(x) = 0$.

$- - - \; 0 \; + + +$ $f''(x)$; 1

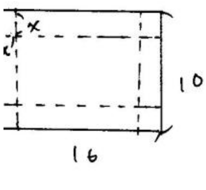
• $f''(x) > 0$ concave upward ($\cup$)

$f''(x) < 0$, concave downward ($\cap$)

concave upward : $(1, \infty)$

concave downward : $(-\infty, 1)$

9. P 317.



体積 $V = (16-2x)(10-2x)x = (16-2x)(10x - 2x^2) = 4x^3 - 32x^2 - 20x^2 + 160x$

$= 4x^3 - 52x^2 + 160x = 4(x^3 - 13x^2 + 40x)$.

$V'(x) = 12x^2 - 104x + 160 = 0$.

$\Rightarrow 4(3x^2 - 26x + 40) = 4(3x - 20)(x - 2) = 0$, $x = \frac{20}{3}, 2$.

<不合>

$x \geq 0$, $16 - 2x \geq 0$ ($x \leq 8$), $10 - 2x \geq 0$ ($x \leq 5$) $\Rightarrow 0 \leq x \leq 5$

$\Rightarrow f(0) = 0$, $f(2) = 144$, $f(5) = 0$ Hence the dimension is $12 \times 6 \times 2$, and the volume is 144