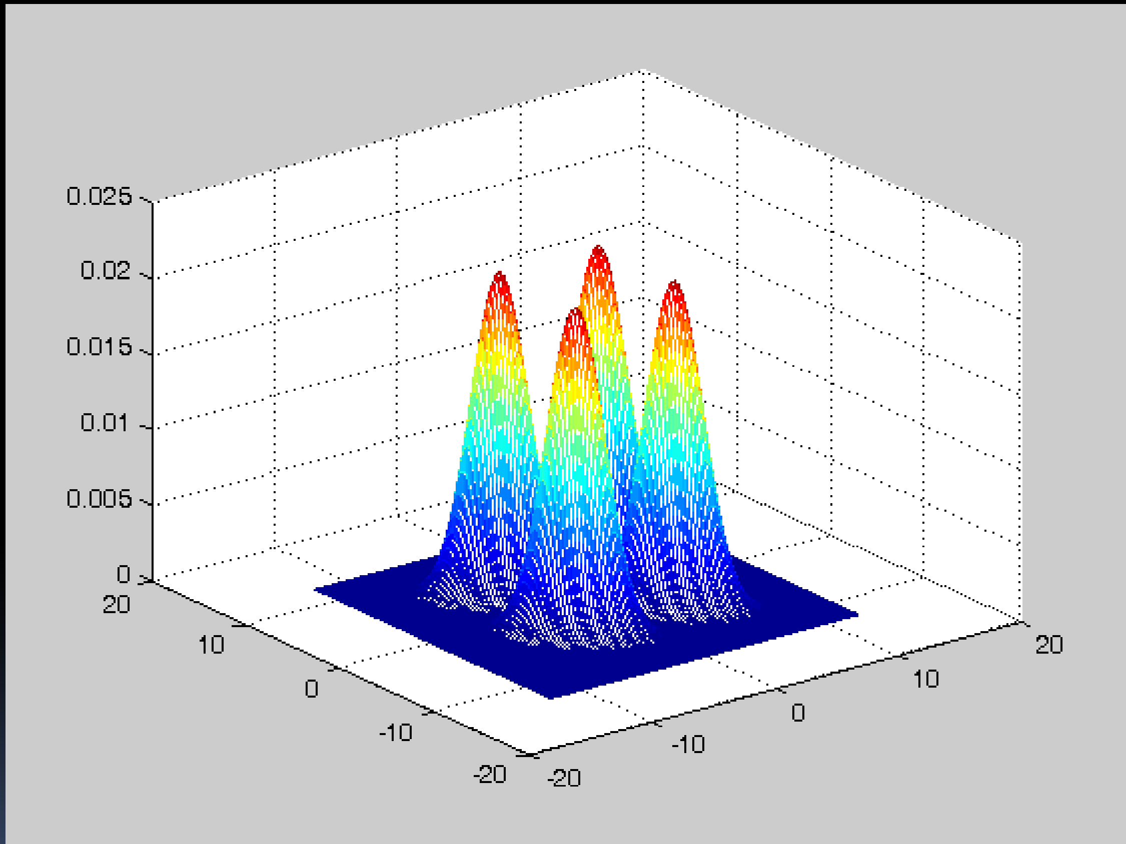
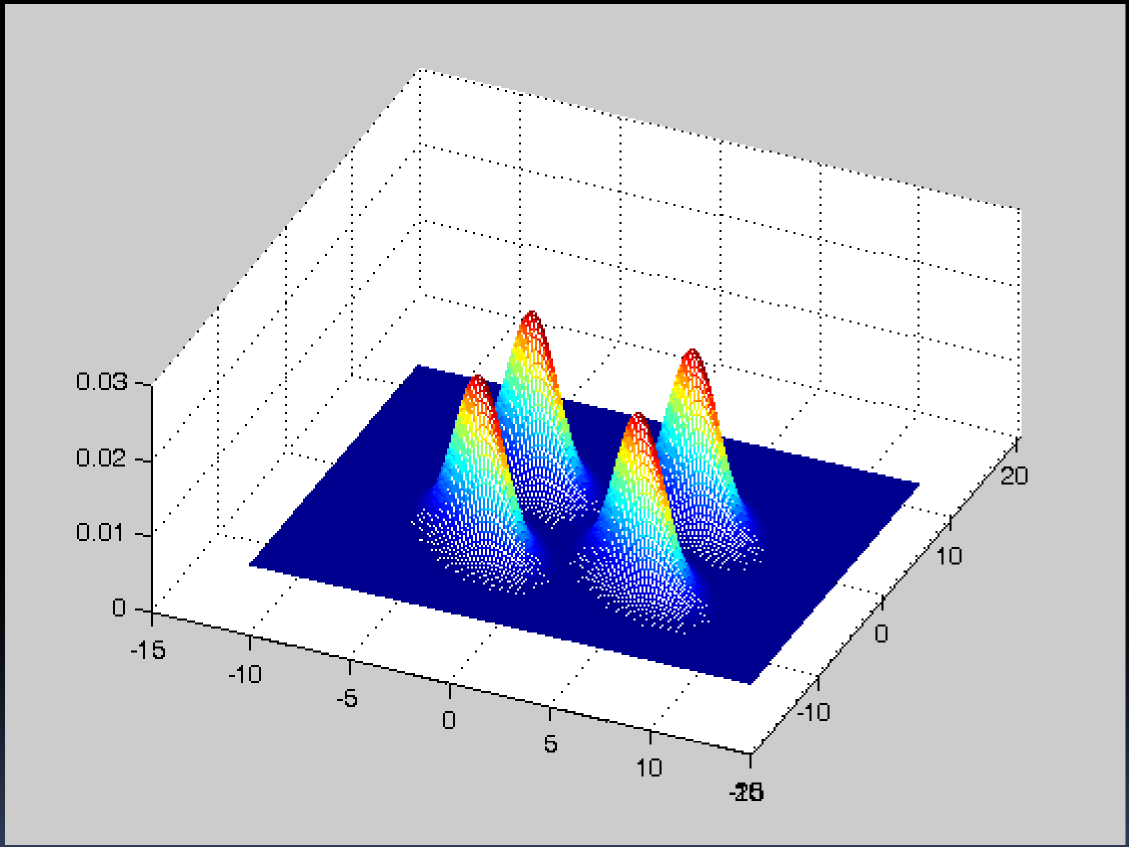
Numerical Integration

Four 2-variate Gaussians



Four 2-variate Gaussians



Gaussian pdf

$$P_k(x) = P(x|y_k, A_k)$$

$$= \frac{1}{(2\pi)^{d/2}\sqrt{|A_k^{-1}|}} \exp\left(\frac{(x - y_k)^t A_k (x - y_k)}{2} \right)$$

Weight sum of Gaussian pdfs

$$q(\mathbf{x}) = \sum_{k} \pi_k p_k(\mathbf{x} \mid \mathbf{A}, \mathbf{y}_k)$$

myfx4.m

```
function out=myfx4(x,y)
global ep;
    A=[0.8 0.2; 0.3 0.75]; d=4;
    u(1,:)=[d d]; u(2,:)=[-d d];
    u(3,:)=[d -d]; u(4,:)=[-d -d];
    A=A'*A;n=length(x);
    c=1/(2*pi*sqrt(det(inv(A))));
    tx=[x;ones(size(x))*y]';
    for i=1:n
        out(i)=0;
         for j=1:4
            out(i)=out(i)+1/4*c*exp(-(tx(i,:)-u(j,:))*A*(tx(i,:)-u(j,:))'/2);
         end
    end
return
```

Plot 4G

plot_4G.m

```
function plot_4G
n=4;
range=n*pi;
x1=-range:0.2:range;
x2=x1;
for i=1:length(x1)
        y_hat=myfx4(x2,x1(i));
        C(i,:)=y_hat;
end
fprintf('max value of fx4:%f\n',max(max(C)));
mesh(x1,x2,C);
```

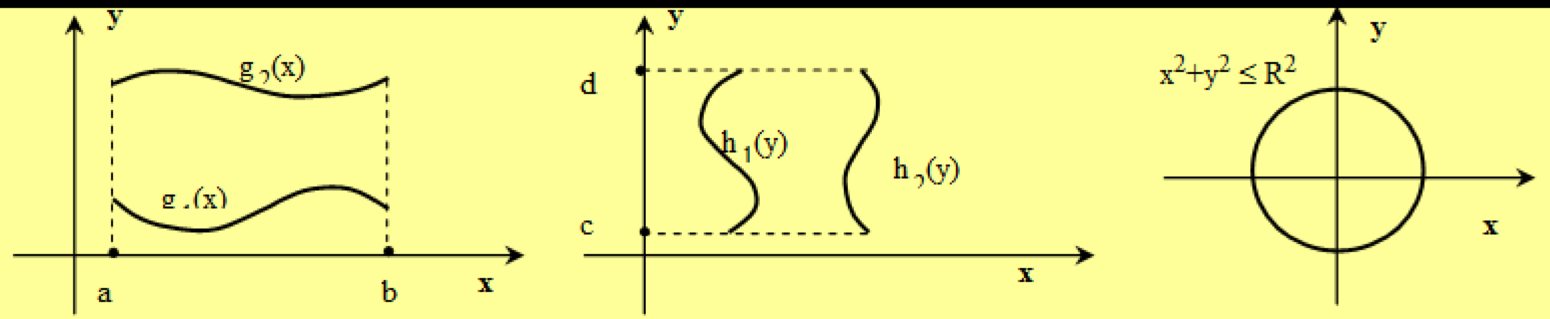
Integration of 4G

demo_int_4G.m

Double integration →

```
function demo_int_4G()
n=4;
xmin = -n*pi;
xmax = n*pi;
ymin = -n*pi;
ymax = n*pi;
result = dblquad(@myfx4,xmin,xmax,ymin,ymax);
fprintf('integration of fx4 over the region :%f \n',result);
```

Numeric calculus for double integrals



Domain normal to x-axis

Domain normal to y-axis

Circular Domain

$$\iint_{D_x} f(x,y)\,ds = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

$$\iint_{D_y} f(x,y)\,ds = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

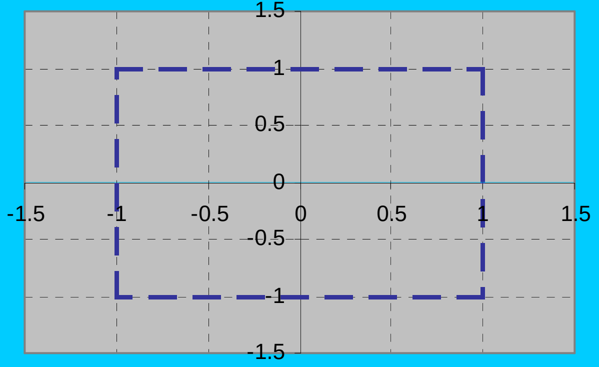
$$\iint_{C} f(x,y)\,ds = \int_0^{2\pi} \int_0^{R} f(\rho\cos(\theta),\rho\sin(\theta))\,\rho\,d\rho\,d\theta$$

Example demo_ex2_4G.m

$$\iint_D f(x,y)ds = \int_{-1}^{1}\int_{-1}^{1} e^{-(x^2+y^2)}\, dy\, dx$$

$$\pi \cdot \left(erf(1)\right)^2$$

Plot of domain D(x, y)



Approx. integral	2.230985141404140
True Integral	2.230985141404130

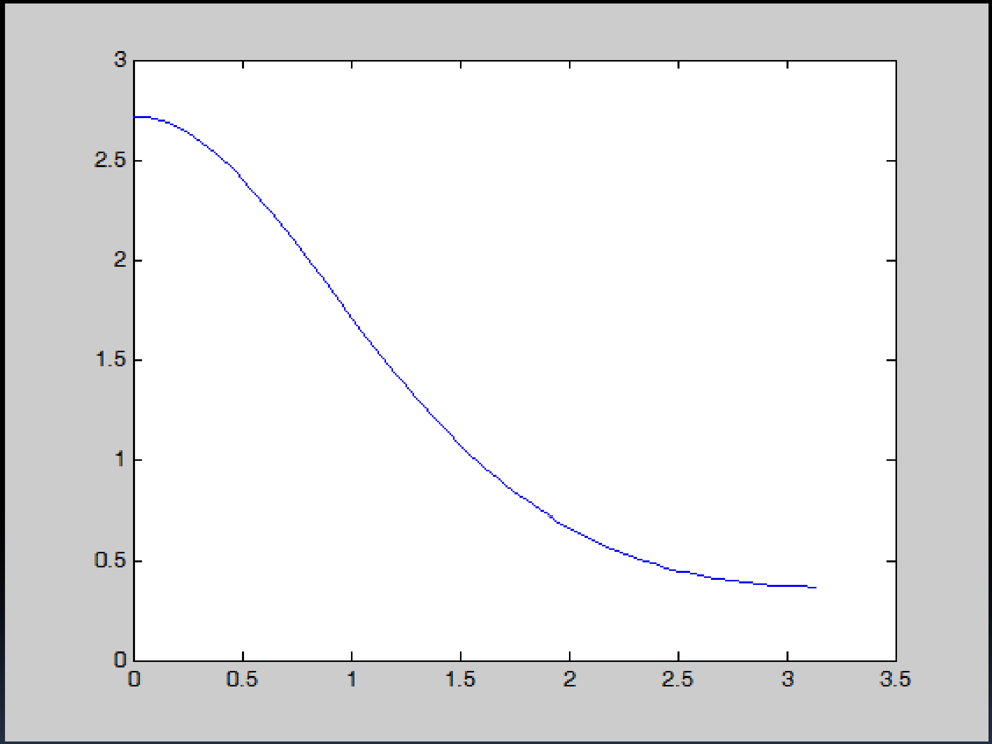
2.230985172585607 by dblquard.m

exp(cos(x))

plot_expcos.m

```
function plot_expcos()
x=linspace(0,pi);
plot(x,fx(x));

function y=fx(x)
    y=exp(cos(x));
return
```



Definite Integration

demo_quad.m

```
function demo_quad()
q = quad(@fx,0,pi);
fprintf('quadrature = %f\n',q);

function y=fx(x)
    y=exp(cos(x));
return
```

```
>> demo_quad
quadrature = 3.977463
```

Symbolic integration

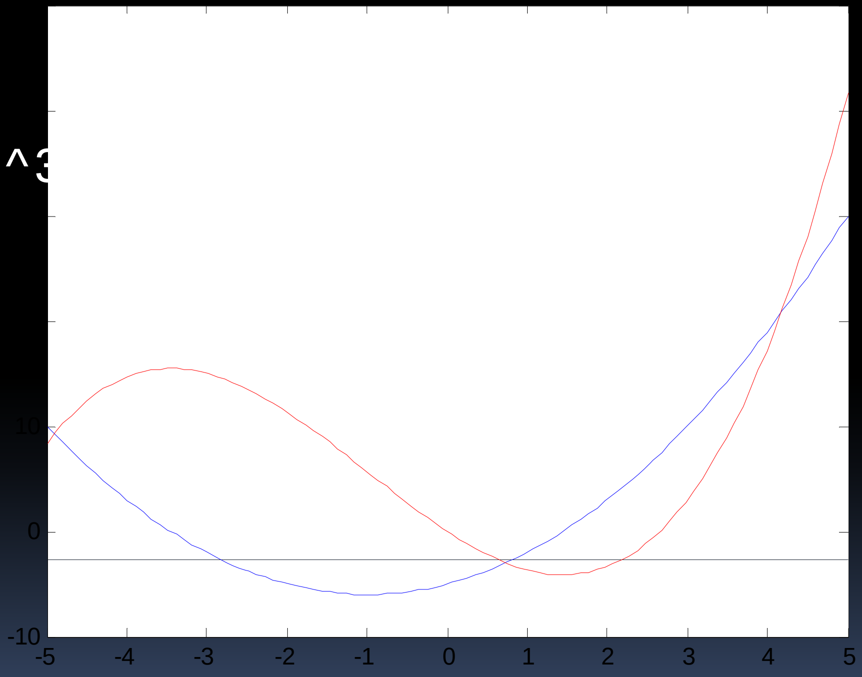
[demo_int.m]

Example

function of x:x.^2+2*x-5

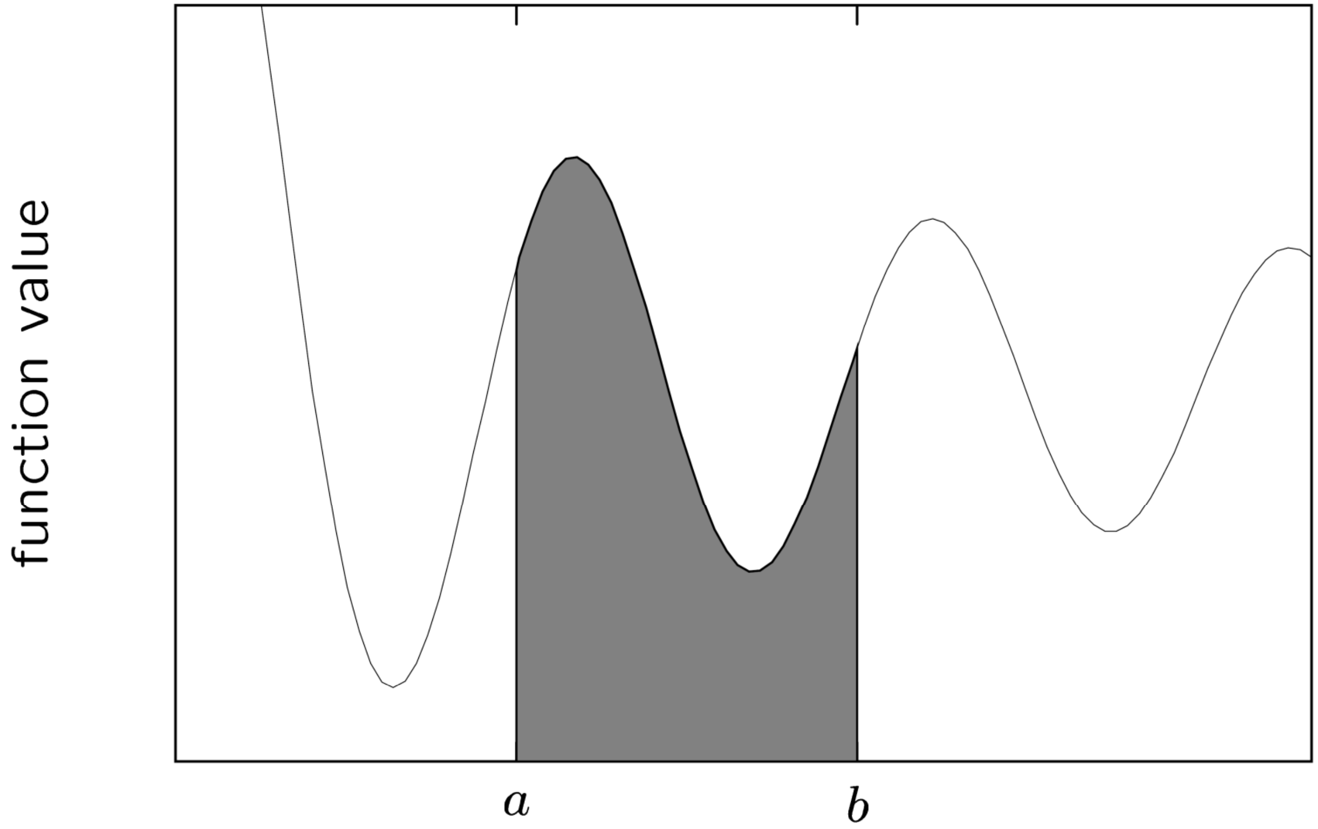
fx1 =

Inline function:
fx1(x) = 1./3.*x.^3



Numerical integration - quadrature

$f(x) \geq 0$ on $[a, b]$ bounded $\Rightarrow \int_a^b f(x)\,dx$ is area under $f(x)$

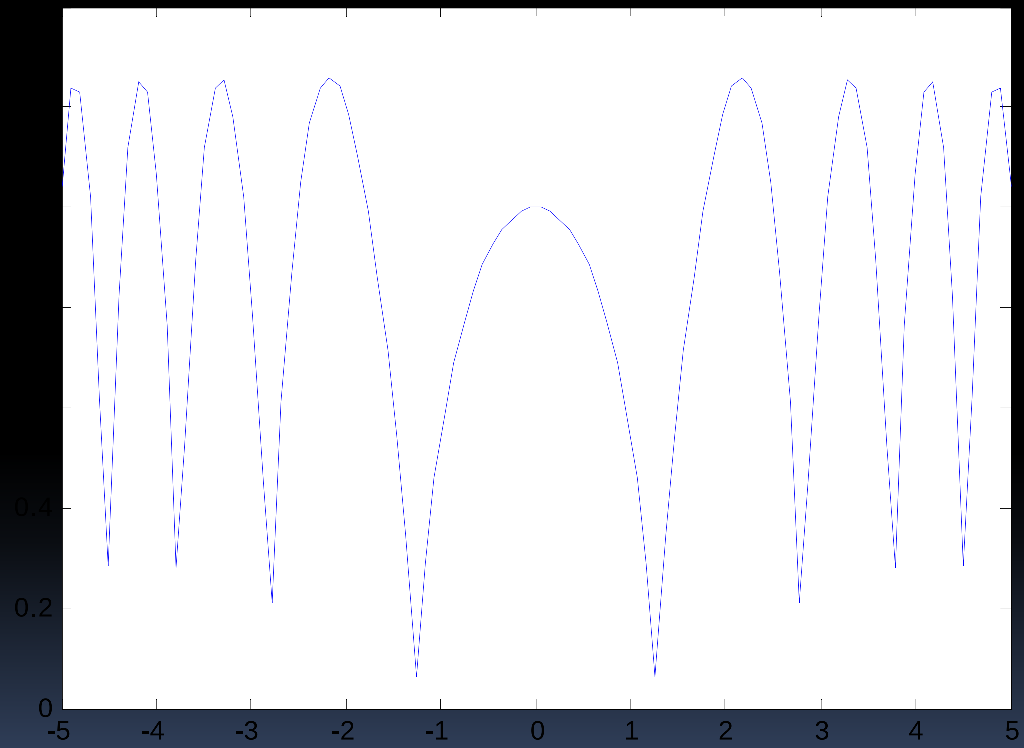


Counter example

$$\int_0^{\frac{\pi}{2}} \left[1 - a^2 \sin^2 \theta\right]^{\frac{1}{3}} d\theta$$

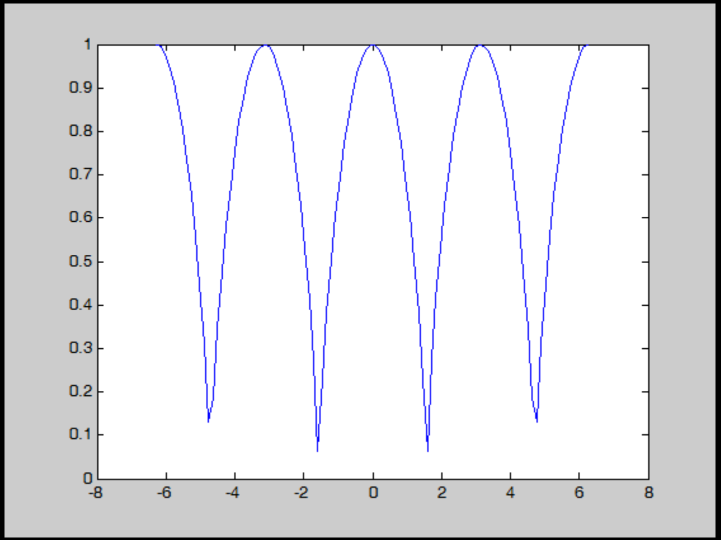
>> demo_int
function of x:(1-sin(x.^2)).^(1/3)
Warning: Explicit integral could not be found.

$$\int_0^{\frac{\pi}{2}} \left[1 - a^2 \sin^2 \theta\right]^{\frac{1}{3}} d\theta$$



plot_sin13.m

```
ss='(1-sin(x).^2).^(1/3)'
fx=inline(ss);
x=linspace(-2*pi,2*pi);
plot(x,fx(x))
```



Numerical integration

demo_quad2.m

```
function demo_quad2()
q = quad(@fx,0,pi);
fprintf('quadrature = %f\n',q);

function y=fx(x)
ss='(1-sin(x).^2).^(1/3)';
f=inline(ss);
y=f(x);
return
```

```
>> demo_quad2
quadrature = 2.240498
```

Mesh

$$P \equiv \{a = x_0 < x_1 < \cdots < x_n = b\}$$

Infima and suprema:

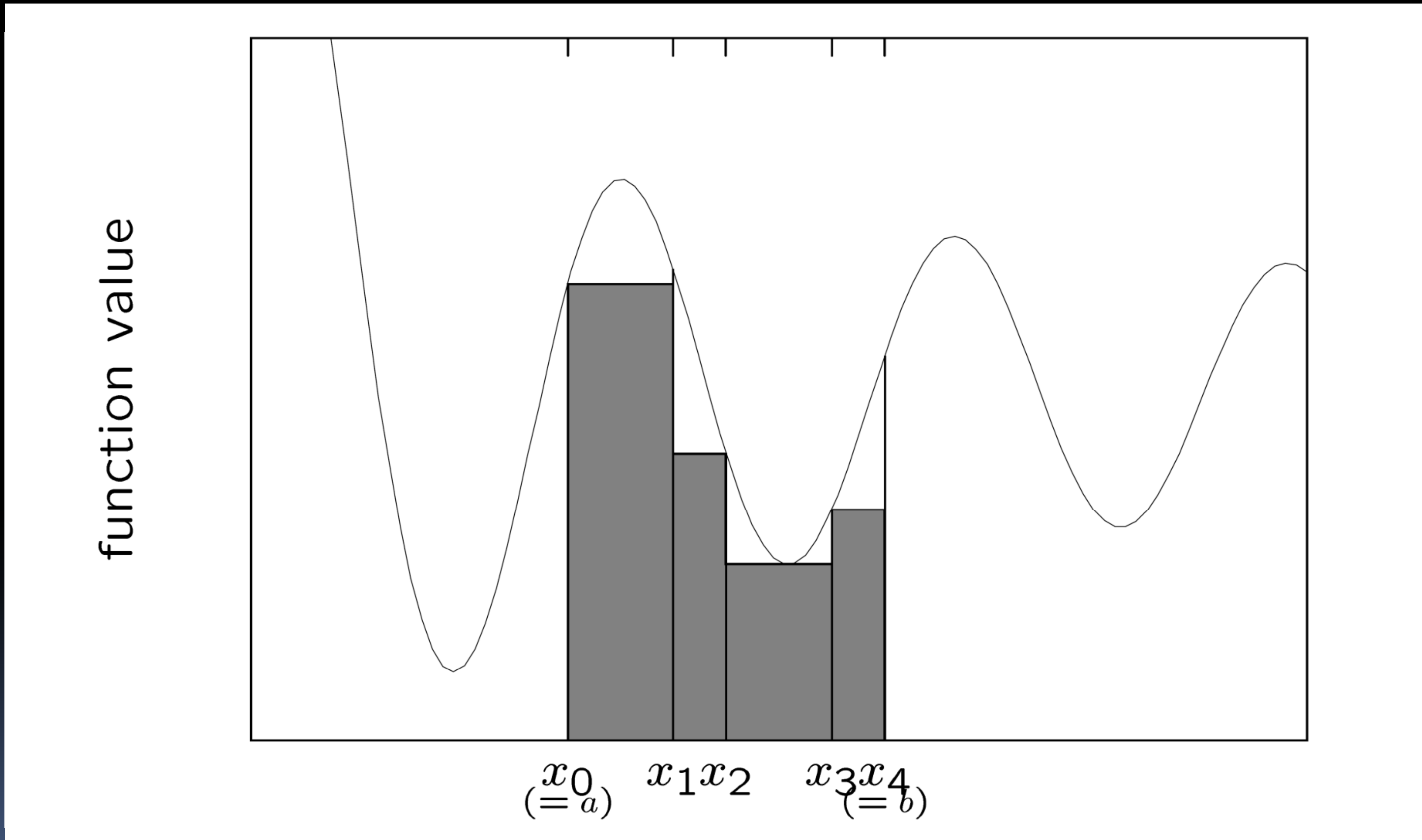
$$m_i \equiv \inf\left\{f(x) : x_i \le x \le x_{i+1}\right\}$$
$$M_i \equiv \sup\left\{f(x) : x_i \le x \le x_{i+1}\right\}$$

Lower and upper sum

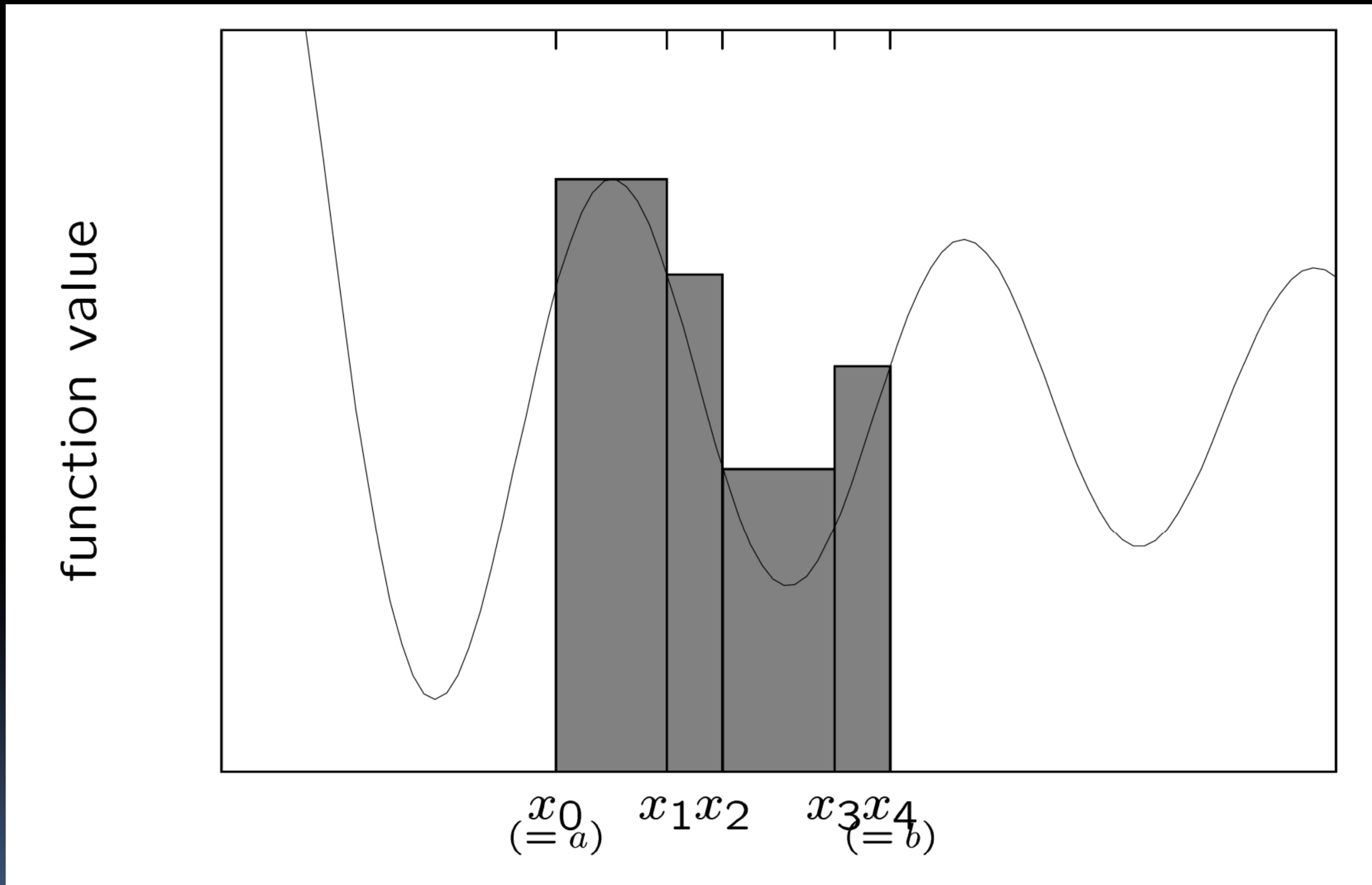
$$L(f;P) \equiv \sum_{i=0}^{n-1} m_i \left(x_{i+1} - x_i\right)$$

$$U(f;P) \equiv \sum_{i=0}^{n-1} M_i \left(x_{i+1} - x_i\right)$$

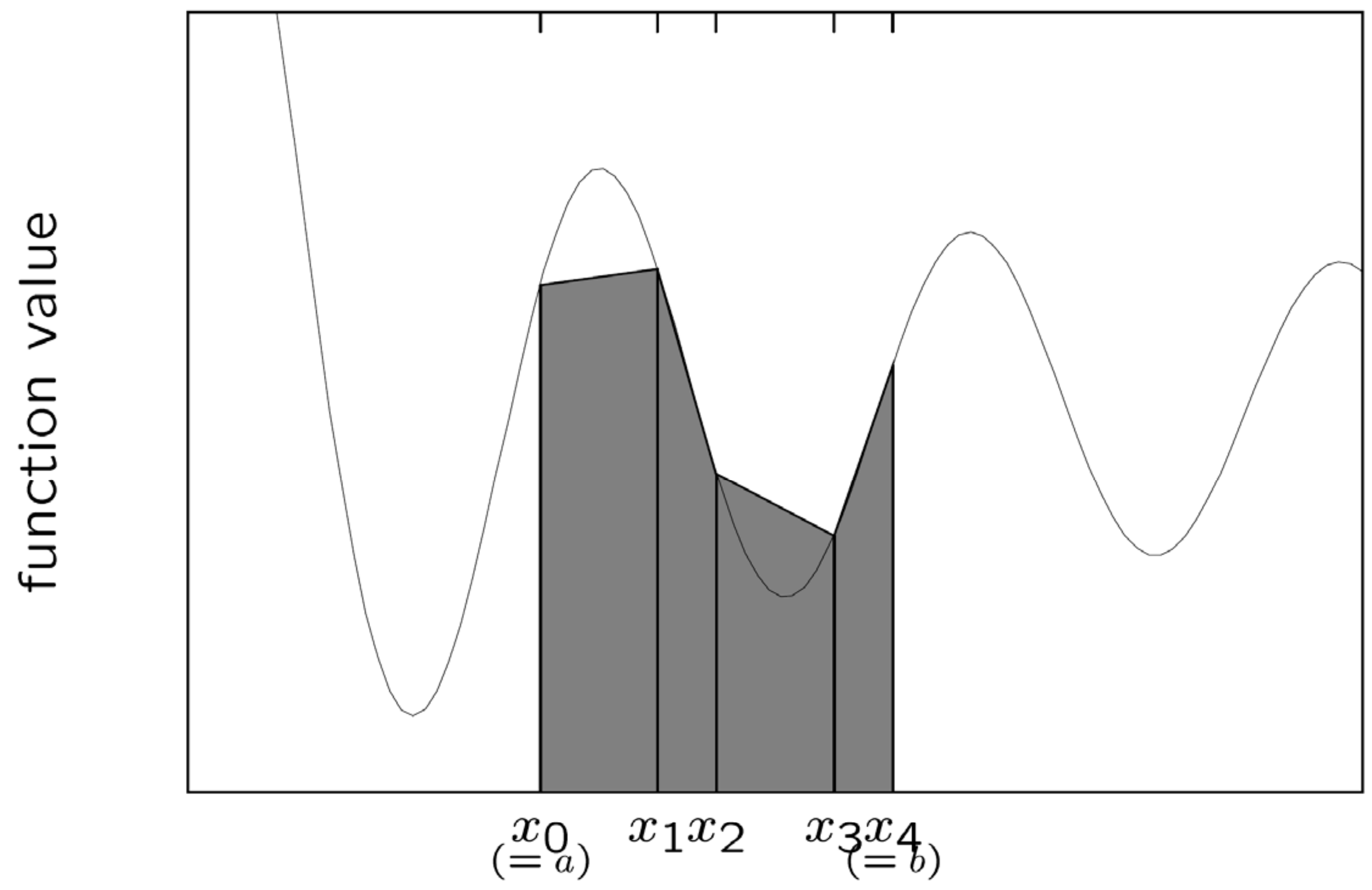
Lower sum : lower bound



Upper sum : upper bound



Composite Trapezoid rule



Uniform mesh

Constant stepsize $h = \frac{b-a}{n}$

$$T(f;P) \equiv h\left\{\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}[f(x_0) + f(x_n)]\right\}$$

Error Analysis

Theorem: $f \in C^2[a,b] \to \exists \xi \in (a,b) \ni$

$$\int_a^b f(x)\,dx - T(f;P) = -\frac{1}{12}(b-a)h^2 f''(\xi) = O\left(h^2\right)$$

Partition size

$$\int_0^{\pi} e^{\cos x}\,dx, \text{ error tolerance} < \frac{1}{2}\times 10^{-3},\ n = ?$$

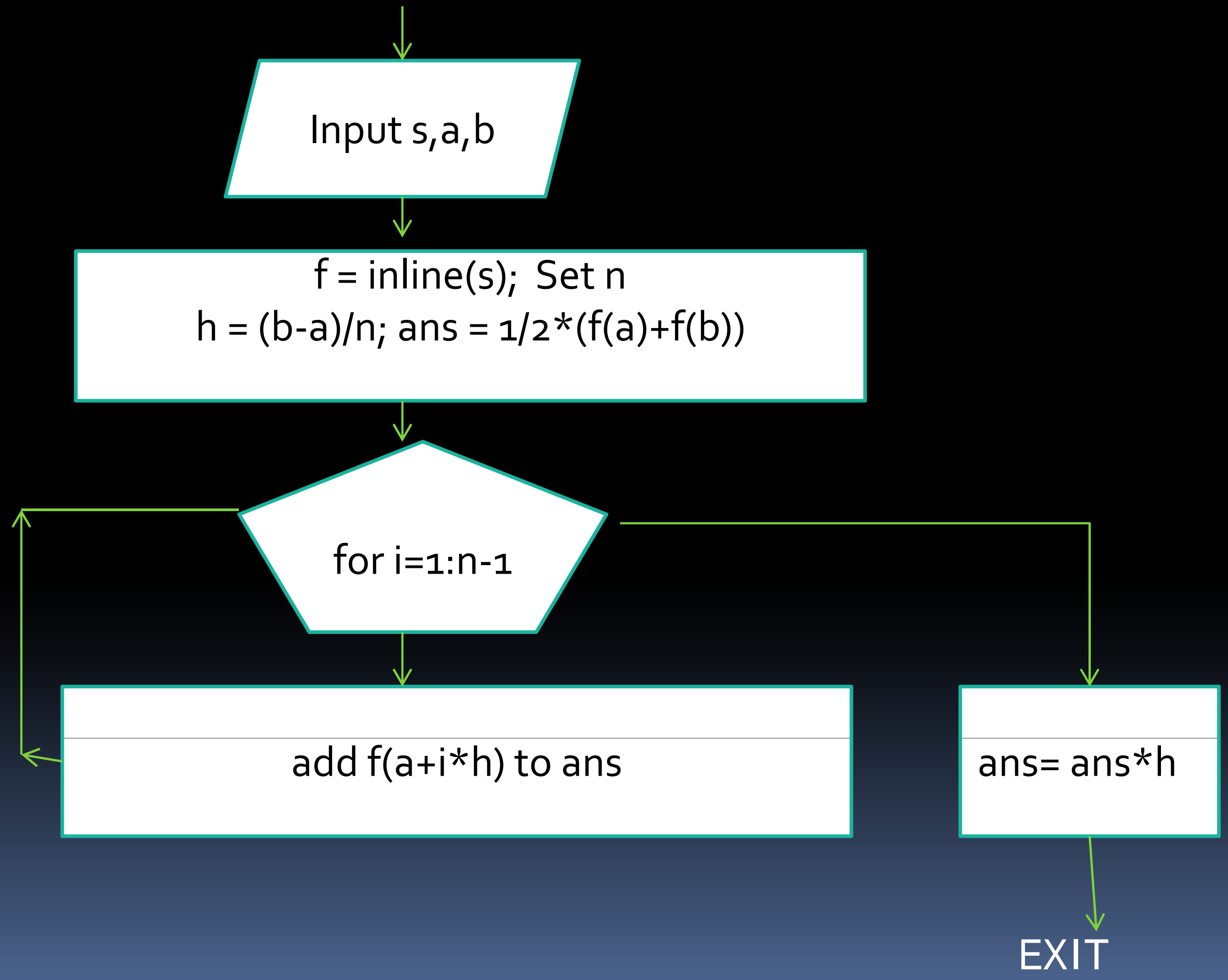
- $f(x) = e^{\cos x} \Rightarrow f'(x) = -e^{\cos x}\sin x \ \ldots\ |f''(x)| \le e$ on $[0,\pi]$
- $\therefore |\text{error}| < \frac{1}{12}\pi(\pi/n)^2 e < \frac{1}{2}\times 10^{-3}$
- $\ldots\ n \ge 119$

Composite Trapezoid rule

- input s, a and b
- f = inline(s); Set n
- h = (b-a)/n; ans = 1/2*(f(a)+f(b))
- for i=1:n-1

  add f(a+i*h) to ans
- ans= ans*h

Flow Chart


Input s,a,b
f = inline(s); Set n
h = (b-a)/n; ans = 1/2*(f(a)+f(b))
for i=1:n-1
add f(a+i*h) to ans
ans= ans*h
EXIT

Simpson rule for numerical integration

If $f \in C^4[a,b]$, then a number ξ in (a, b) exists with

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b)\right] - \frac{f^{(4)}(\xi)}{2880}(b-a)^5$$

Exercise

- Draw a flow chart to illustrate integration by the composite Trapezoid rule
- Implement the composite Trapezoid rule for numerical integration, including flow chart and Matlab codes
- Test your matlab function with the following integration

$$\int_0^{\pi} f(x)$$

f(x)=exp(cos(x))

- * Test your matlab function with definite integration of the weight sum of four Gaussian pdfs
- * Compare your results with those obtained by using quad.m