Simpson rule
Composite Simpson rule

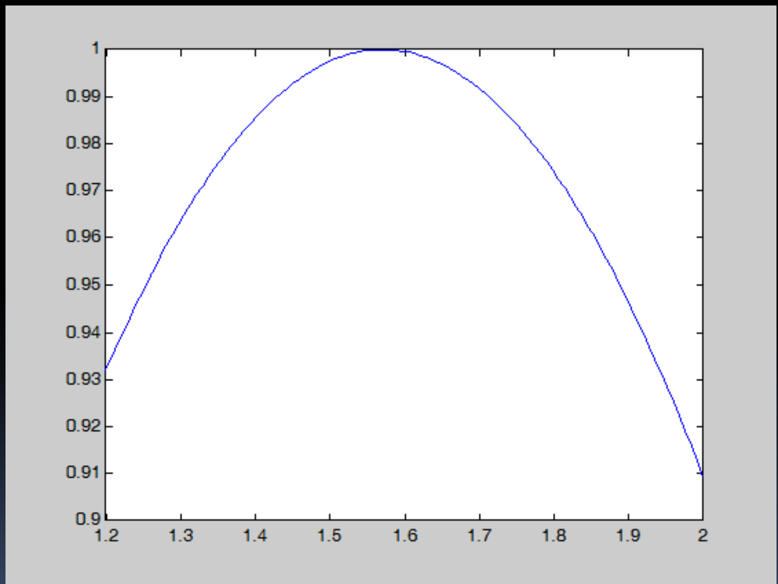
Simpson rule for numerical integration

If $f \in C^4[a,b]$, then a number ξ in (a, b) exists with

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b) \right] - \frac{f^{(4)}(\xi)}{2880}(b-a)^5$$

y=sin(x)   for x within [1.2 2]

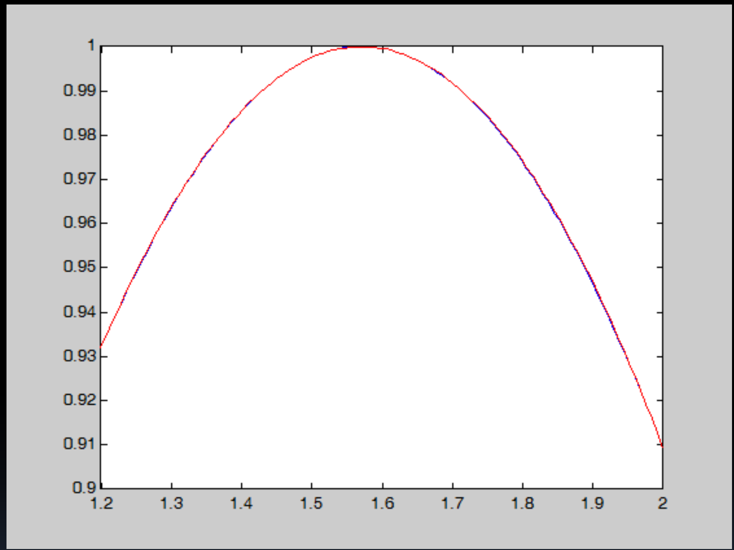
```
a=1.2;b=2;
x=linspace(a,b);plot(x,sin(x));hold on
```



Quadratic polynomial

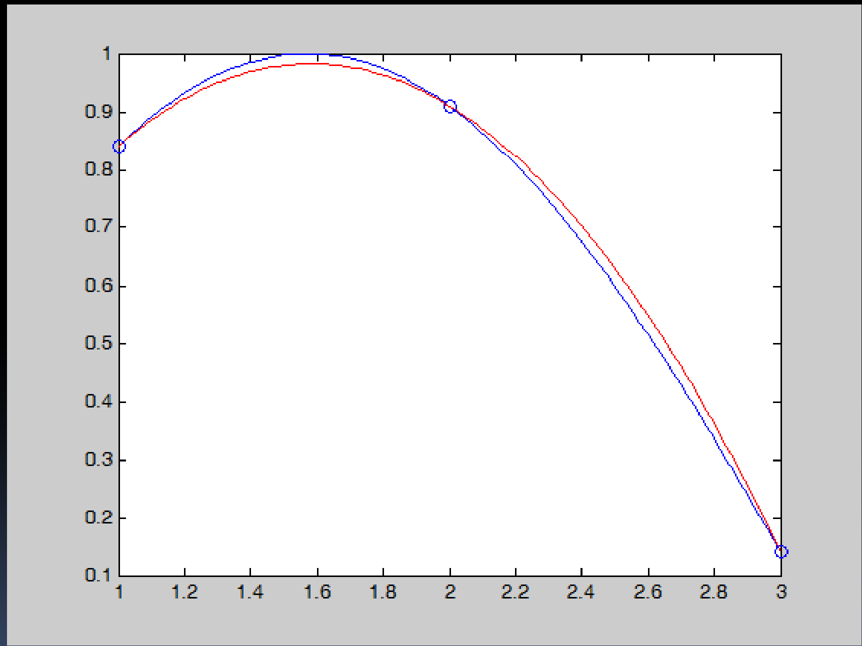
- Approximate sin using a quadratic polynomial

```
c=0.5*(a+b);
x=[a b c]; y=sin(x);
p=polyfit(x,y,2);
z=linspace(a,b);
plot(z,polyval(p,z) ,'r')
```



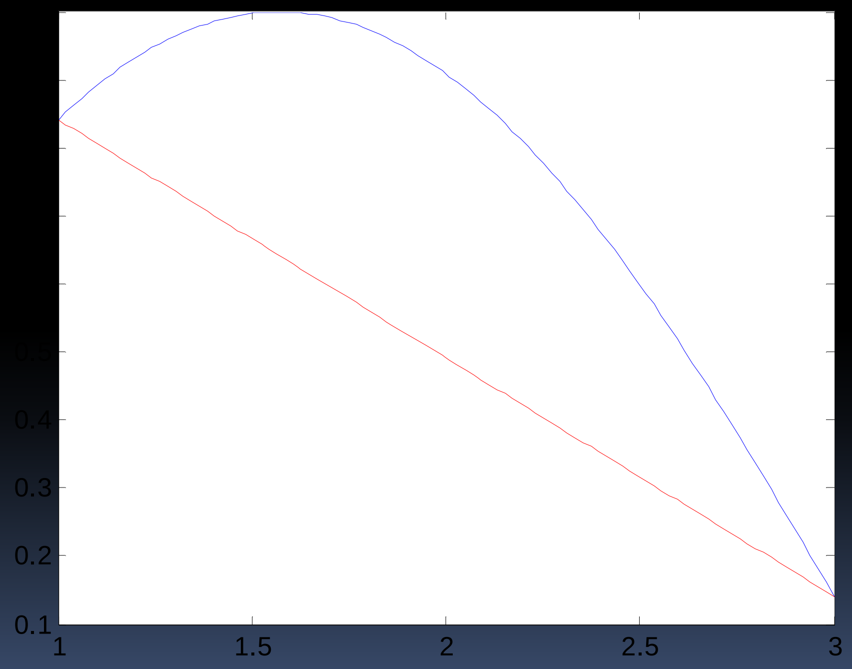
Approximate sin(x) within [1 3] by a quadratic polynomial

```
a=1;b=3;
x=linspace(a,b);plot(x,sin(x))
hold on;
c=0.5*(a+b);
x=[a b c]; y=sin(x);
p=polyfit(x,y,2);
z=linspace(a,b);

plot(z,polyval(p,z) ,'r')
```



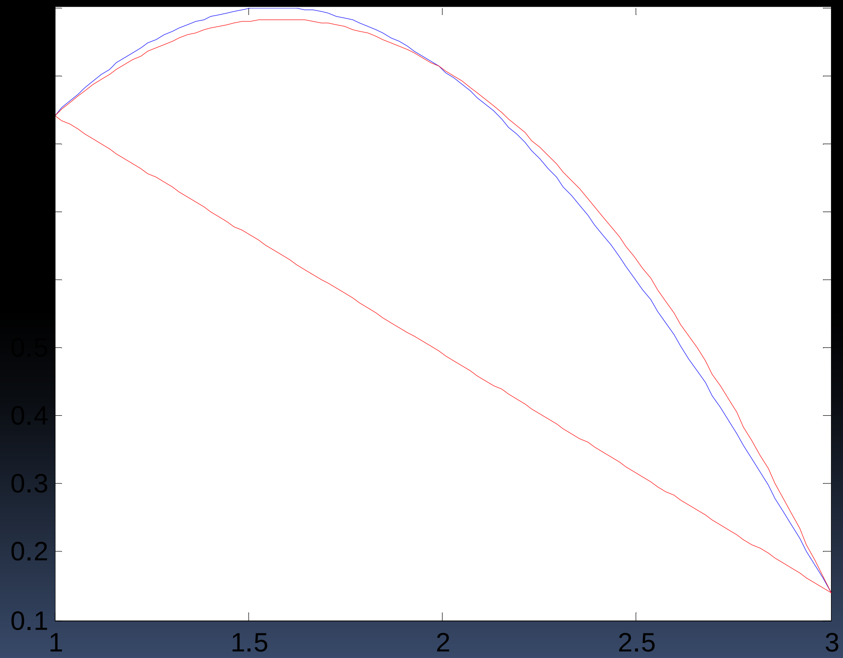
Approximate sin(x) within [1 3] by a line

```
a=1;b=3;
x=linspace(a,b);plot(x,sin(x))
hold on;
c=0.5*(a+b);
x=[a b ]; y=sin(x);
p=polyfit(x,y,1);
z=linspace(a,b);

plot(z,polyval(p,z) ,'r')
```



Strategy I : Apply Trapezoid rule to calculate area under a line
Strategy II : Apply Simpson rule to calculate area under a second order polynomial

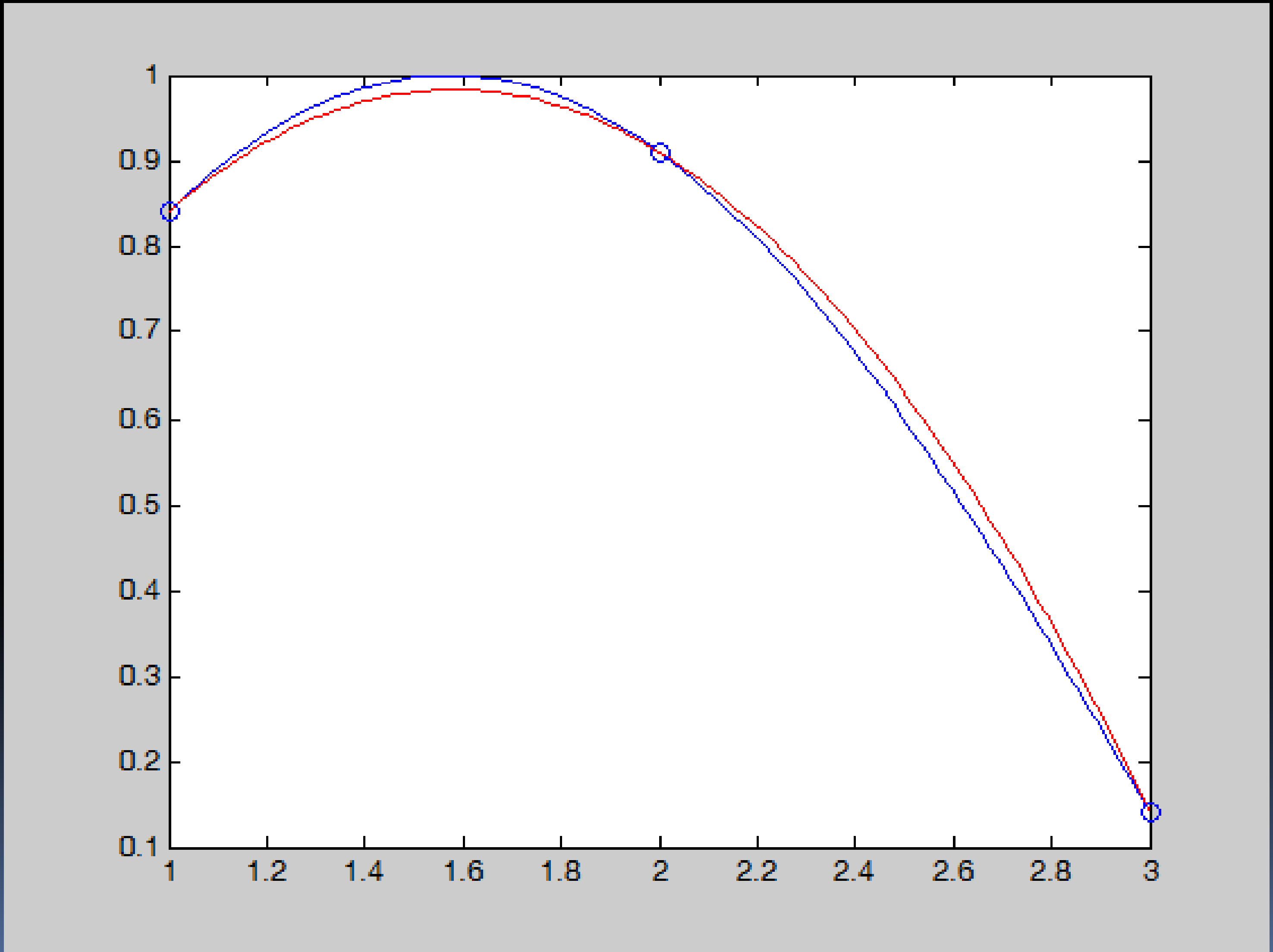
Strategy II is more accurate and general than strategy I, since a line is a special case of quadratic polynomial



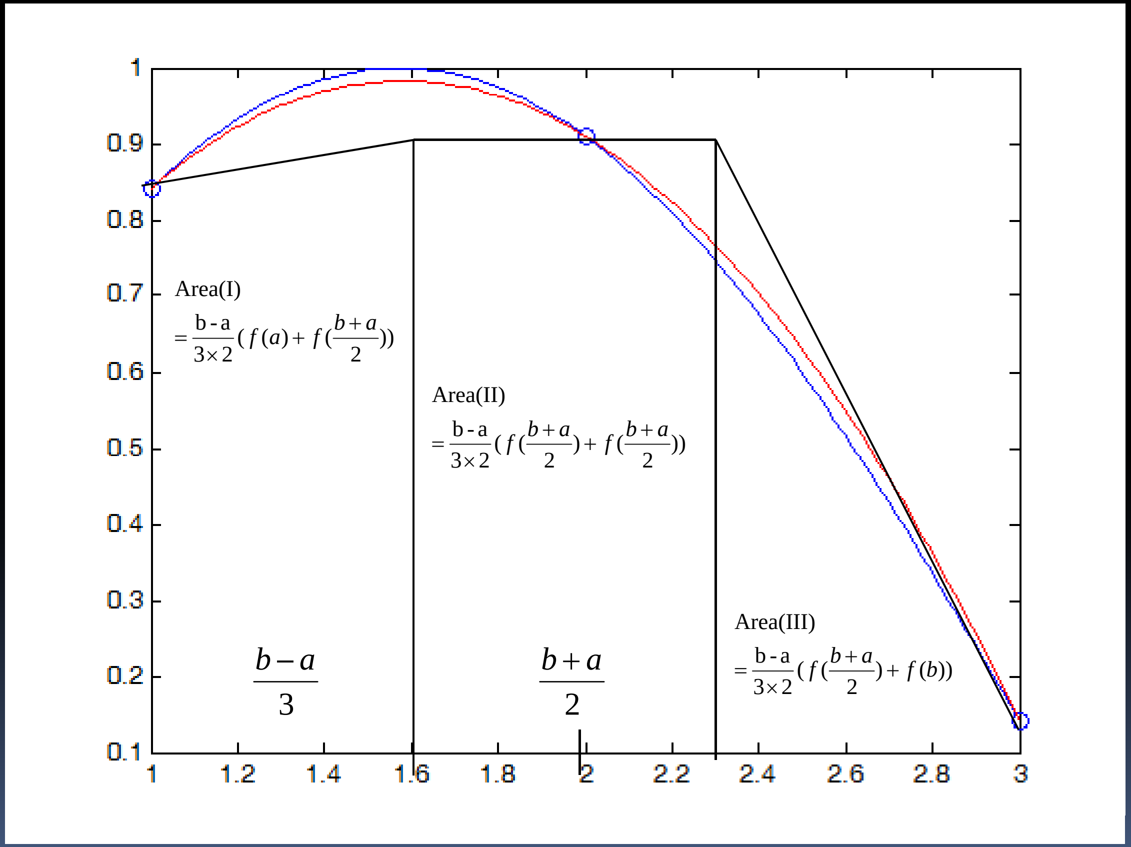
Simpson rule

- Original task: integration of f(x) within [a,b]
- c=(a+b)/2
- Numerical task
  - Approximate f(x) within [a,b] by a quadratic polynomial, p(x)
  - Integration of p(x) within [a,b]

Blue: f(x)=sin(x) within [1,3]
Red: a quadratic polynomial that pass (1,sin(1)), (2,sin(2)),(3,sin(3))



- The area under a quadratic polynomial is a sum of area I, II and III



- The area under a quadratic polynomial is a sum of area I, II and III

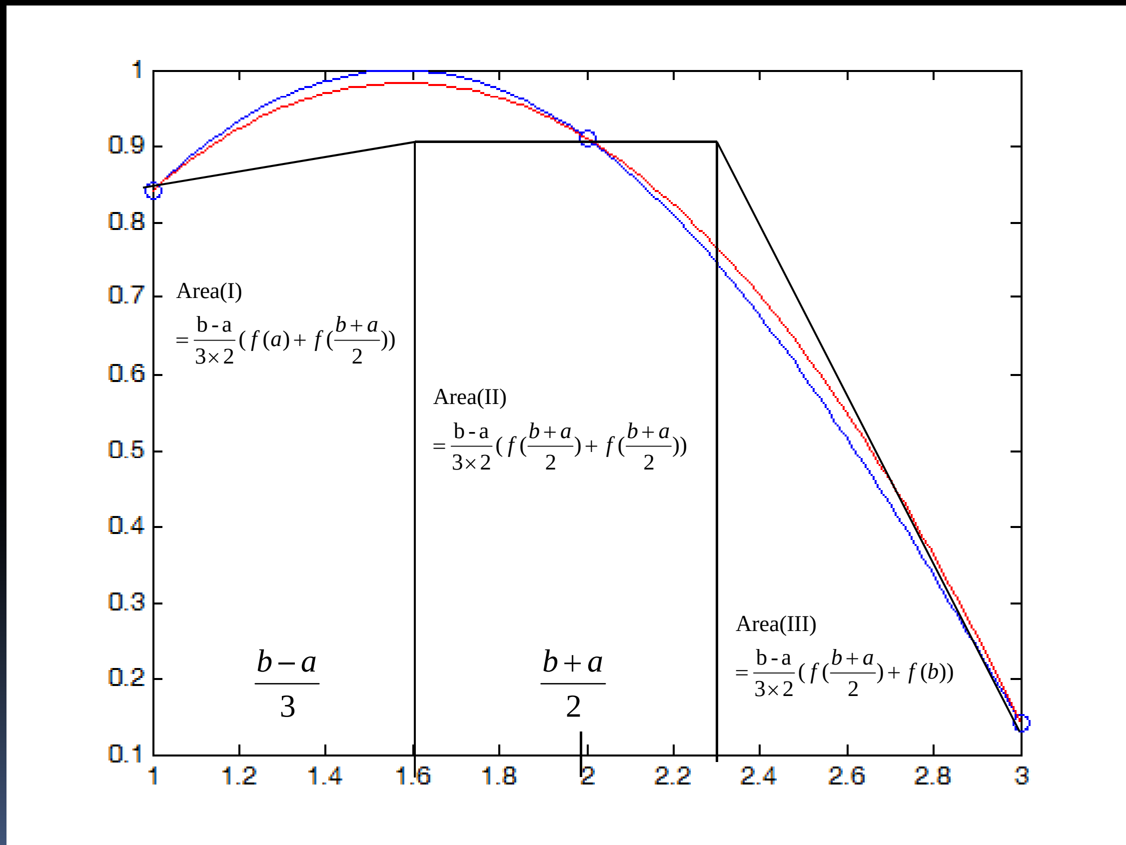
- Partition [a,b] to three equal-size intervals,
  and use the high of the middle point c to produce three Trapezoids

- Use the composite Trapezoid rule to determine area I, II and III
  h/2*(f(a)+f(c) +f(c)+f(c)+ f(c)+f(b))

- substitute h=(b-a)/3   and c=(a+b)/2
- area I + II + III = (b-a)/6 *(f(a)+4*f((a+b)/2) +f(b))

$$\text{Area(I)} + \text{Area(II)} + \text{Area(III)}$$

$$= \frac{b - a}{3 \times 2}\left(f(a) + 4f\left(\frac{b+a}{2}\right) + f(b)\right)$$



$p(x) \in P_2$

$p(x)$ passes $(a, f(a))$ $(b, f(b))$ $(\frac{a+b}{2}, f(\frac{a+b}{2}))$

Show that

$$\int_a^b p(x)dx = \frac{b-a}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b) \right]$$

$$(a, f(a))$$

$$(b, f(b))$$

$$(c, f(c)), c = \frac{a+b}{2}$$

$$p(x) = \left[f(a)\frac{(x-b)(x-c)}{(a-b)(a-c)} + f(b)\frac{(x-a)(x-c)}{(b-a)(b-c)} + f(c)\frac{(x-a)(x-b)}{(c-a)(c-b)} \right]$$

$$proof : p(a) = f(a), p(b) = f(b), p(c) = f(c)$$

$$\int_a^b p(x)dx = \int_a^b \left[f(a)\frac{(x-b)(x-c)}{(a-b)(a-c)} + f(b)\frac{(x-a)(x-c)}{(b-a)(b-c)} + f(c)\frac{(x-a)(x-b)}{(c-a)(c-b)} \right] dx$$

$$= f(a)\int_a^b \frac{(x-b)(x-c)}{(a-b)(a-c)}dx + f(b)\int_a^b \frac{(x-a)(x-c)}{(b-a)(b-c)}dx + f(c)\int_a^b \frac{(x-a)(x-b)}{(c-a)(c-b)}dx$$

$b \to a+2h$

$c \to a+h$

$$= \frac{h}{3}(f(a)+4f(c)+f(b))$$

$$= \frac{(b-a)}{6}(f(a)+4f(c)+f(b))$$

$$\int_a^b \frac{(x-b)(x-c)}{(a-b)(a-c)}dx = \int_a^{a+2h} \frac{(x-a-2h)(x-a-h)}{(a-b)(a-c)}dx = h\int_0^2 \frac{(t-2)(t-1)}{(0-2)(0-1)}dt = h\frac{1}{3}$$

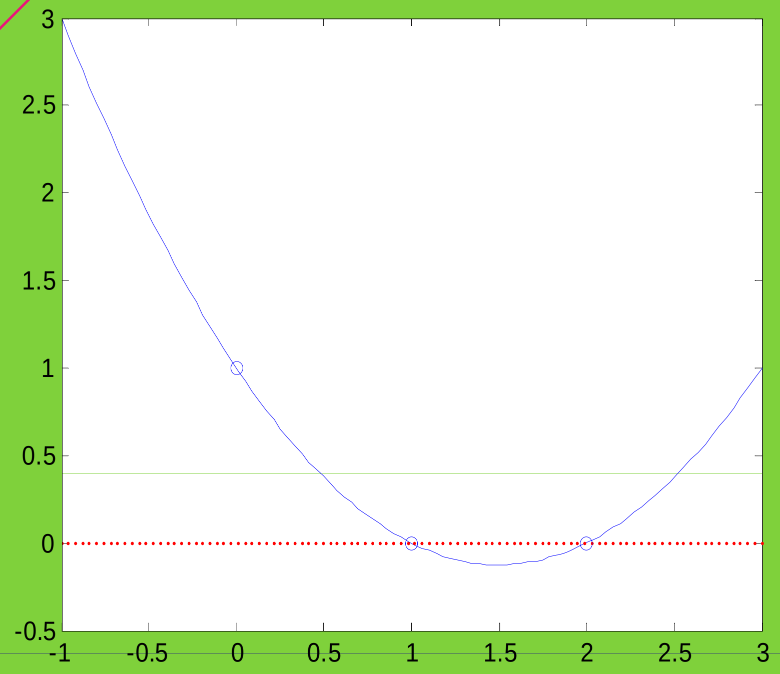
$$\int_a^b \frac{(x-a)(x-b)}{(c-a)(c-b)}dx = h\int_0^2 \frac{(t-0)(t-2)}{(1-2)(1-0)}dt = h\frac{4}{3}$$

$$\int_a^b \frac{(x-a)(x-c)}{(b-a)(b-c)}dx = h\int_0^2 \frac{(t-0)(t-1)}{(2-1)(2-0)}dt = h\frac{1}{3}$$

$$\int_a^b \frac{(x-b)(x-c)}{(a-b)(a-c)}dx = \int_a^{a+2h} \frac{(x-a-2h)(x-a-h)}{(a-b)(a-c)}dx$$

$$= \int_0^{2h} \frac{(t-2h)(t-h)}{(0-2h)(0-h)}dt = h\int_0^{2} \frac{(t-2)(t-0)}{(0-2)(0-1)}dt$$

Shift Lagrange polynomial
Rescale Lagrange polynomial



Shortcut to Simpson 1/3 rule

$$p(x) = Ax^2 + Bx + C$$

$$p \in P_2$$

$$\int_a^b p(x)dx = Q(I) + Q(II) + Q(III)$$

$$= \frac{b-a}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b))$$

1. Partition [a,b] to three equal intervals
2. Draw three trapezoids, Q(I), Q(II) and Q(III)
3. Calculate the sum of areas of the three trapezoids

Proof

$$p(x) = \frac{1}{(a-b)^2}\left[2f(a)(x-b)(x-c) + 2f(b)(x-a)(x-c) - 4f(c)(x-a)(x-b)\right]$$

$$= Ax^2 + Bx + C$$

$$p \in P_2$$

$$\int_a^b p(x)dx = \int_a^b (Ax^2 + Bx + C)dx$$

$$= \left[\frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\right]_a^b$$

$$= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

Composite Simpson rule

- Partition [a,b] into n interval
- Integrate each interval by Simpson rule
- h=(b-a)/2n

$$\int_a^b f(x)dx$$

$$= \sum_{i=0}^{n-1} \int_{a+2ih}^{a+(2i+2)h} f(x)dx$$

Apply Simpson sule to each interval

$$\int_a^b p(x)dx$$

$$= \frac{b-a}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b))$$

$$\int_{a+2ih}^{a+(2i+2)h} f(x)dx$$

$$= \frac{h}{3}(f(a+2ih) + 4f(a+(2i+1)h) + f(a+(2i+2)h))$$

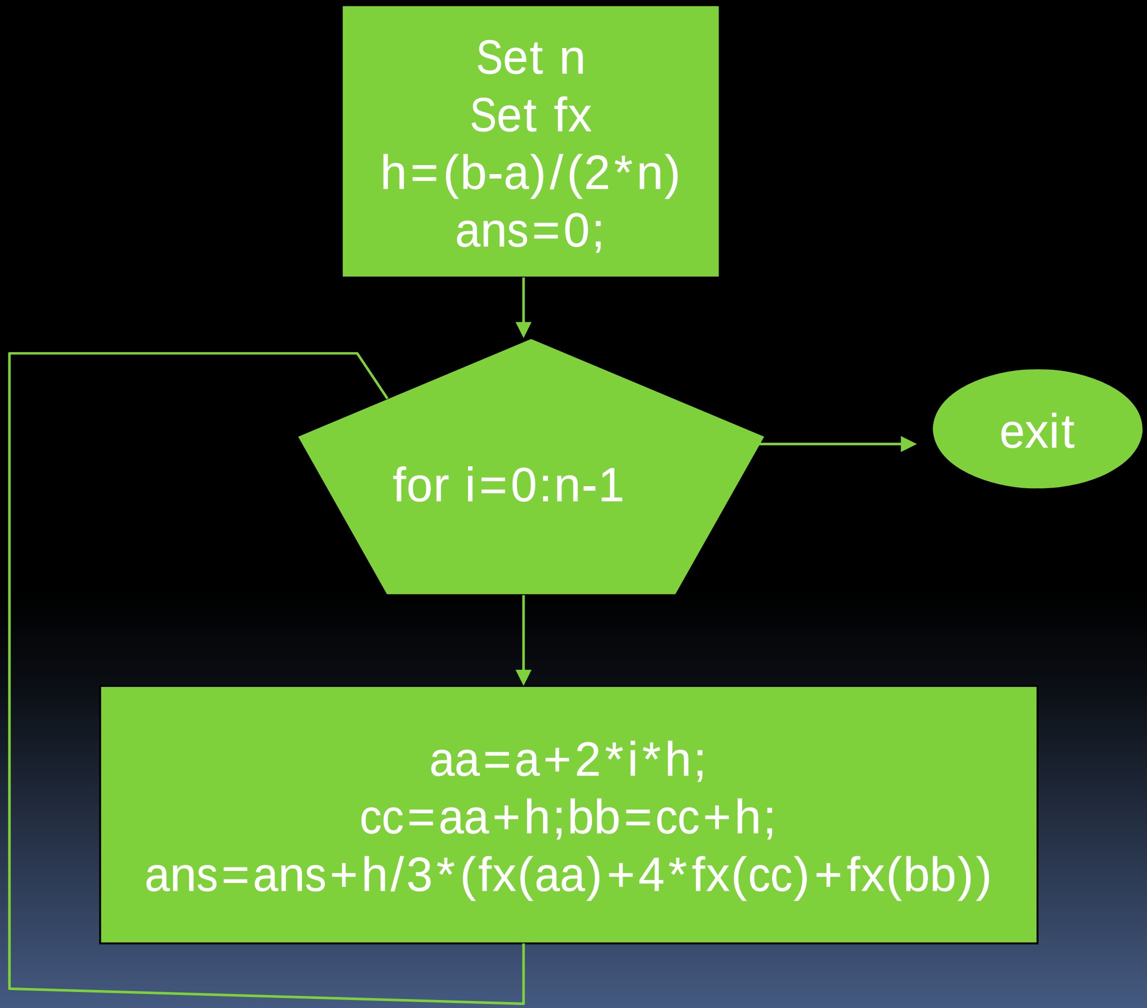
Composite Simpson rule

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{a+2ih}^{a+(2i+2)h} f(x)dx$$

$$\approx \frac{h}{3} \sum_{i=0}^{n-1} (f(a+2ih) + 4f(a+(2i+1)h) + f(a+(2i+2)h))$$

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{a+2ih}^{a+(2i+2)h} f(x)dx$$

$$\approx \frac{h}{3} \sum_{i=0}^{n-1} \left(f(a+2ih) + 4f(a+(2i+1)h) + f(a+(2i+2)h) \right)$$



Composite Simpson rule

$$x_k = a + jh, j = 0,1,\ldots,2n$$ h=(b-a)/2n

$$\int_a^b f(x)dx$$

$$\approx \frac{h}{3}\sum_{i=0}^{n-1}(f(a+2ih)+4f(a+(2i+1)h)+f(a+(2i+2)h))$$

$$=\frac{h}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots+f(x_{2n})\right]$$

$$=\frac{h}{3}(f(a)+f(b))+\frac{2h}{3}\sum_{k=1}^{n-1}f(x_{2k})+\frac{4h}{3}\sum_{k=1}^{n-1}f(x_{2k-1})$$

Composite Simpson rule

$$x_k = a + jh,\ j = 0,1,\ldots,2n$$ h=(b-a)/2n

$$\int_a^b f(x)dx$$

$$\approx \frac{h}{3}(f(a)+f(b)) + \frac{2h}{3}\sum_{k=1}^{n-1} f(x_{2k}) + \frac{4h}{3}\sum_{k=1}^{n-1} f(x_{2k-1})$$

Simpson's Rule for Numerical Integration