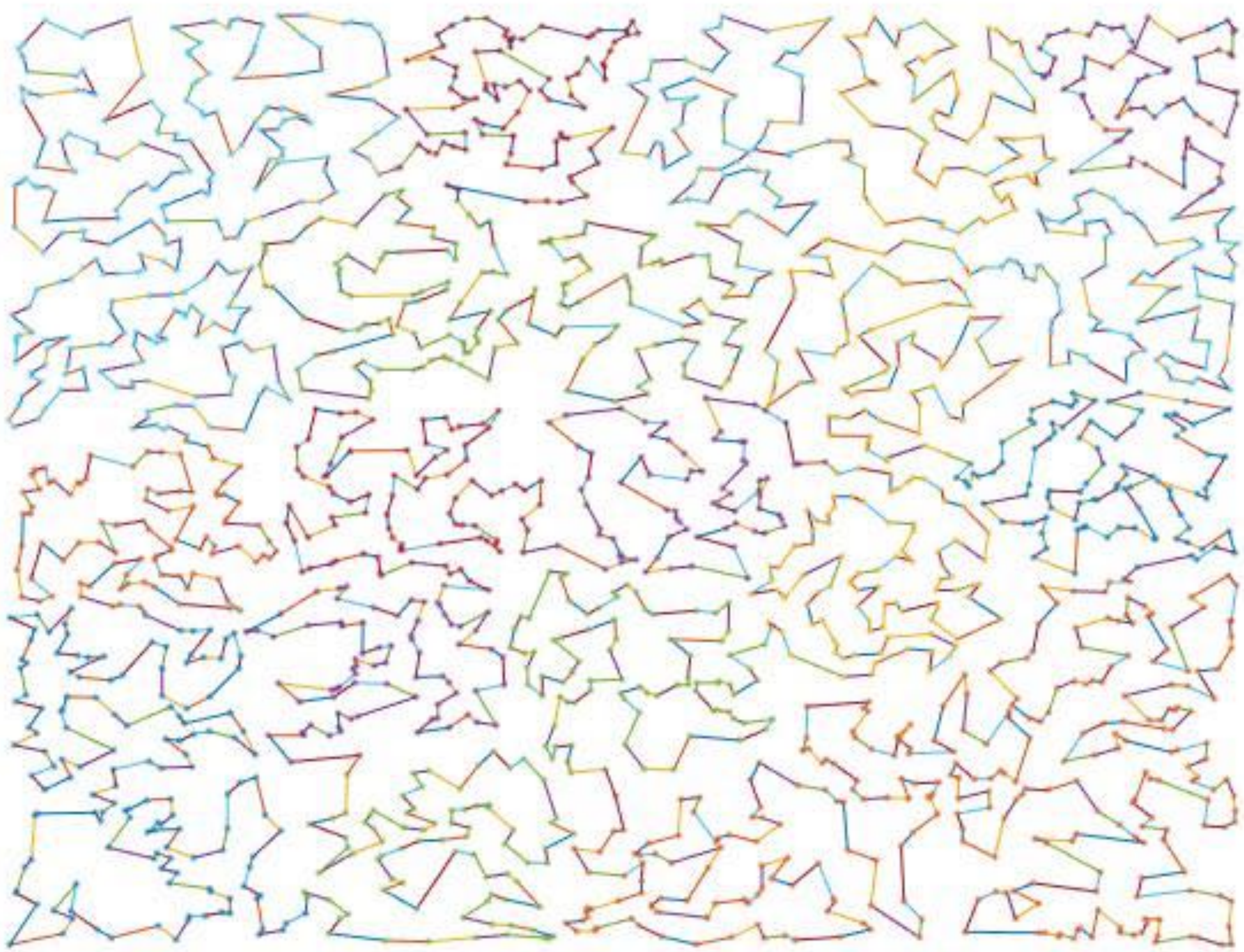
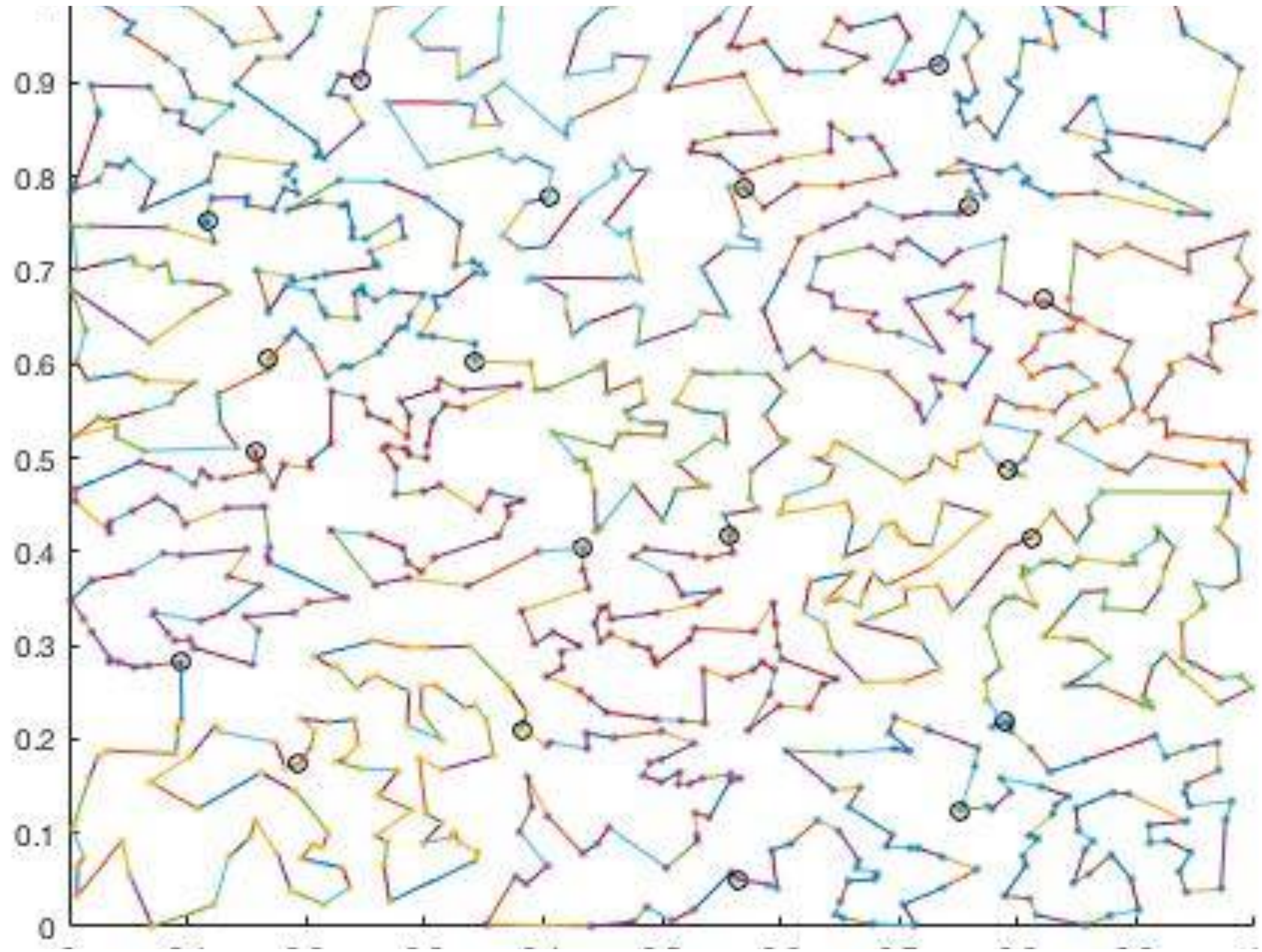
Outline

- Matrix multiplication
- Reduced echelon form
- Gauss-Jordan Elimination
- Forward elimination and backward substitution

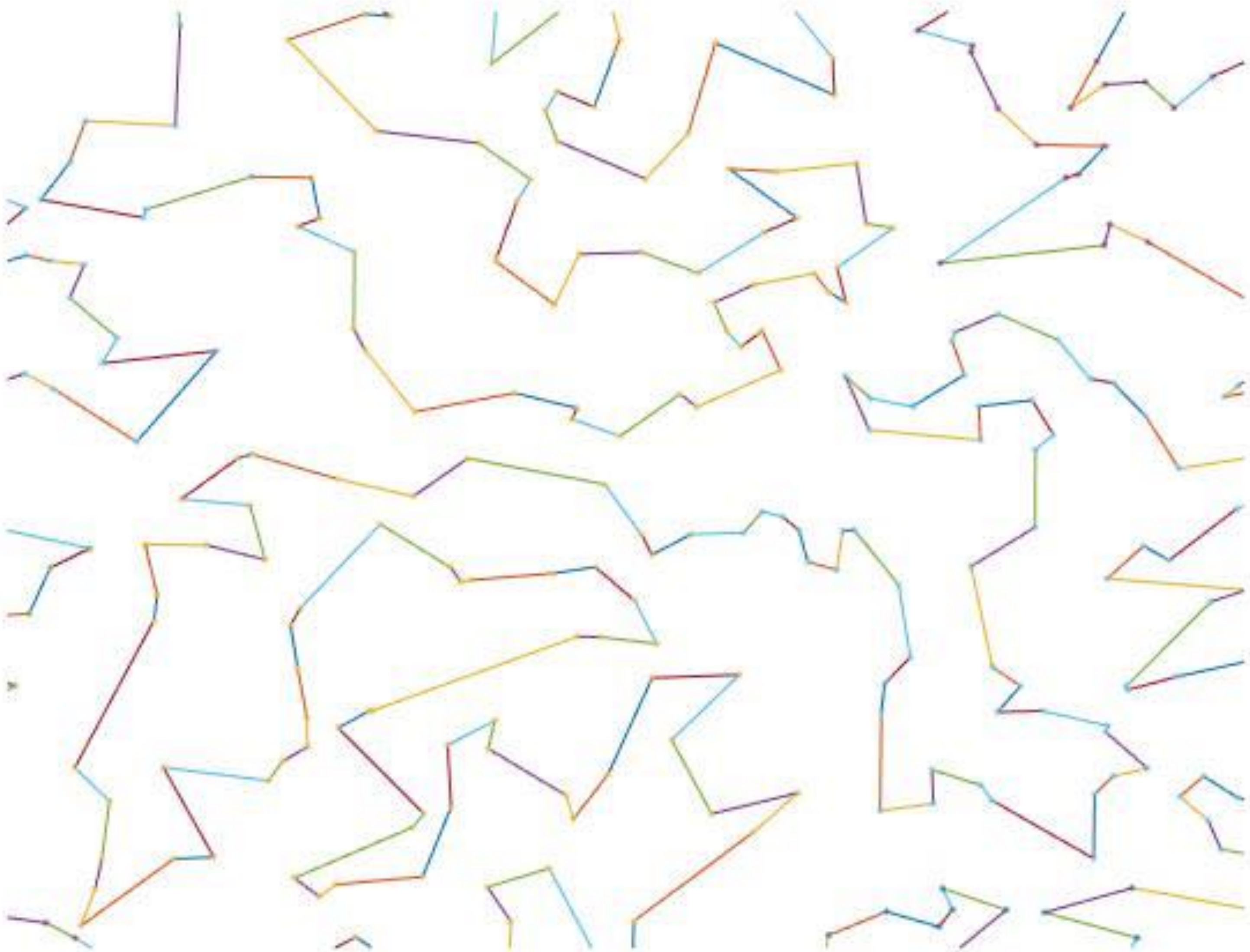
TSP-2000



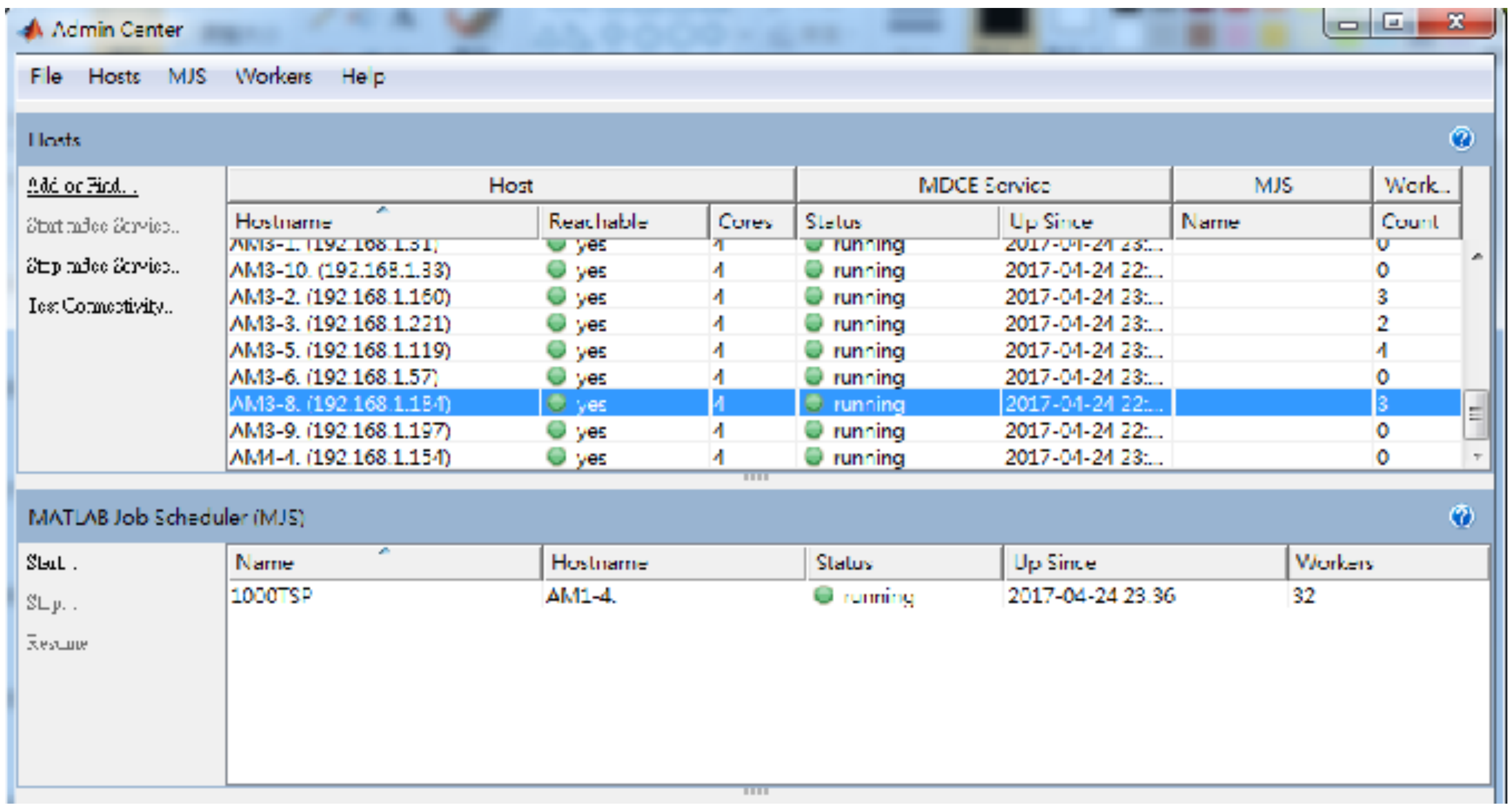
TSP



Collective decisions



32-workers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Hosts
Add or Find...
Start mdce Service...
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AM3-2. (192.168.1.160) yes 4 running 2017-04-24 23:... 3
AM3-3. (192.168.1.221) yes 4 running 2017-04-24 23:... 2
AM3-5. (192.168.1.119) yes 4 running 2017-04-24 23:... 4
AM3-6. (192.168.1.57) yes 4 running 2017-04-24 23:... 0
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AM3-9. (192.168.1.197) yes 4 running 2017-04-24 22:... 0
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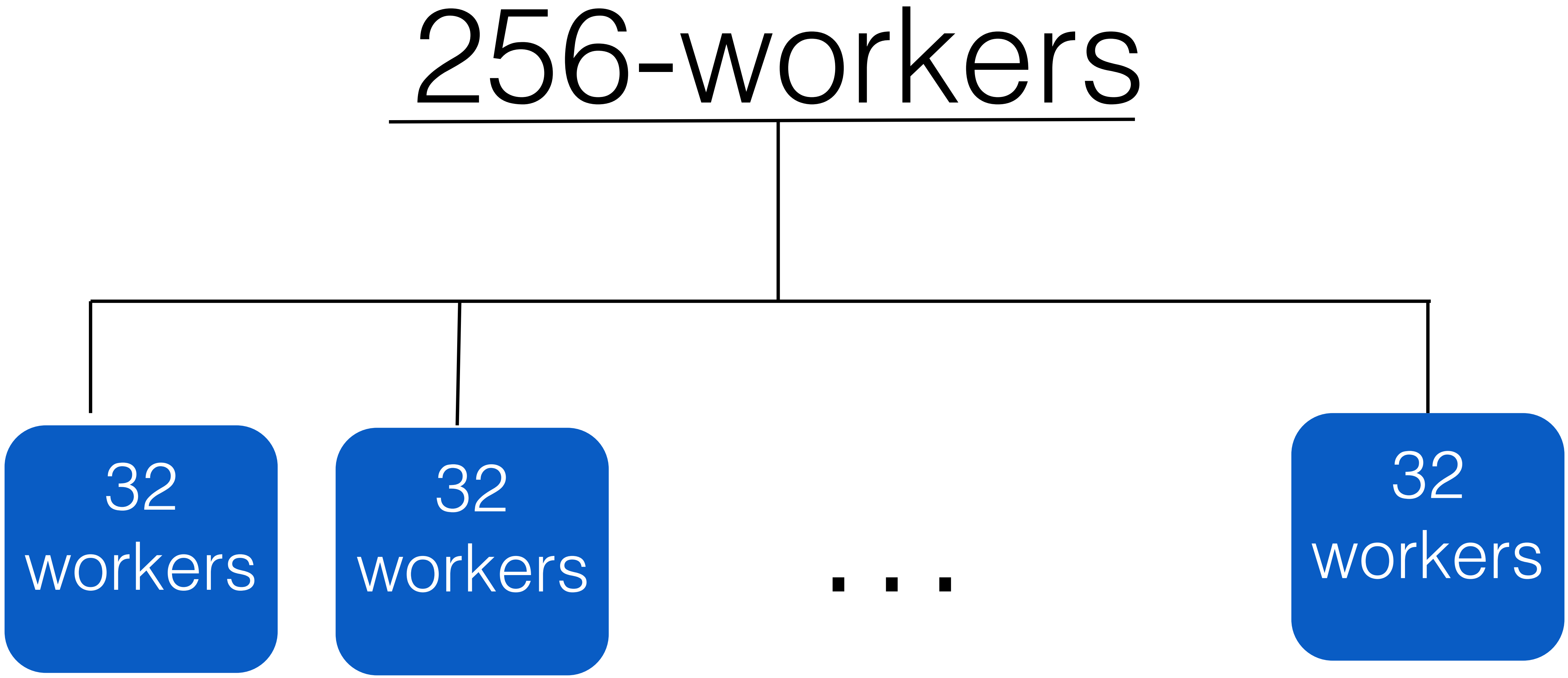
256-workers
32
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workers

Example

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 & 1 \\ 3 & -2 & 6 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 & 1 \\ 3 & -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \end{bmatrix} & \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \end{bmatrix} \\ \begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \end{bmatrix} & \begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (1\times 5)+(3\times 3) & (1\times 0)+(3\times(-2)) & (1\times 1)+(3\times 6) \\ (2\times 5)+(2\times 3) & (2\times 0)+(0\times(-2)) & (2\times 1)+(0\times 6) \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -6 & 19 \\ 10 & 0 & 2 \end{bmatrix}.$$

Example 3

Let $C = AB$ for the following matrices A and B. Determine the element c_{23} of C.

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -7 & 3 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

c_{23} is the element in row 2, column 3 of C. It will be the product of row 2 of A and column 3 of B. We get

Solution

$$c_{23} = \begin{bmatrix} -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (-3 \times 2) + (4 \times 1) = -2$$

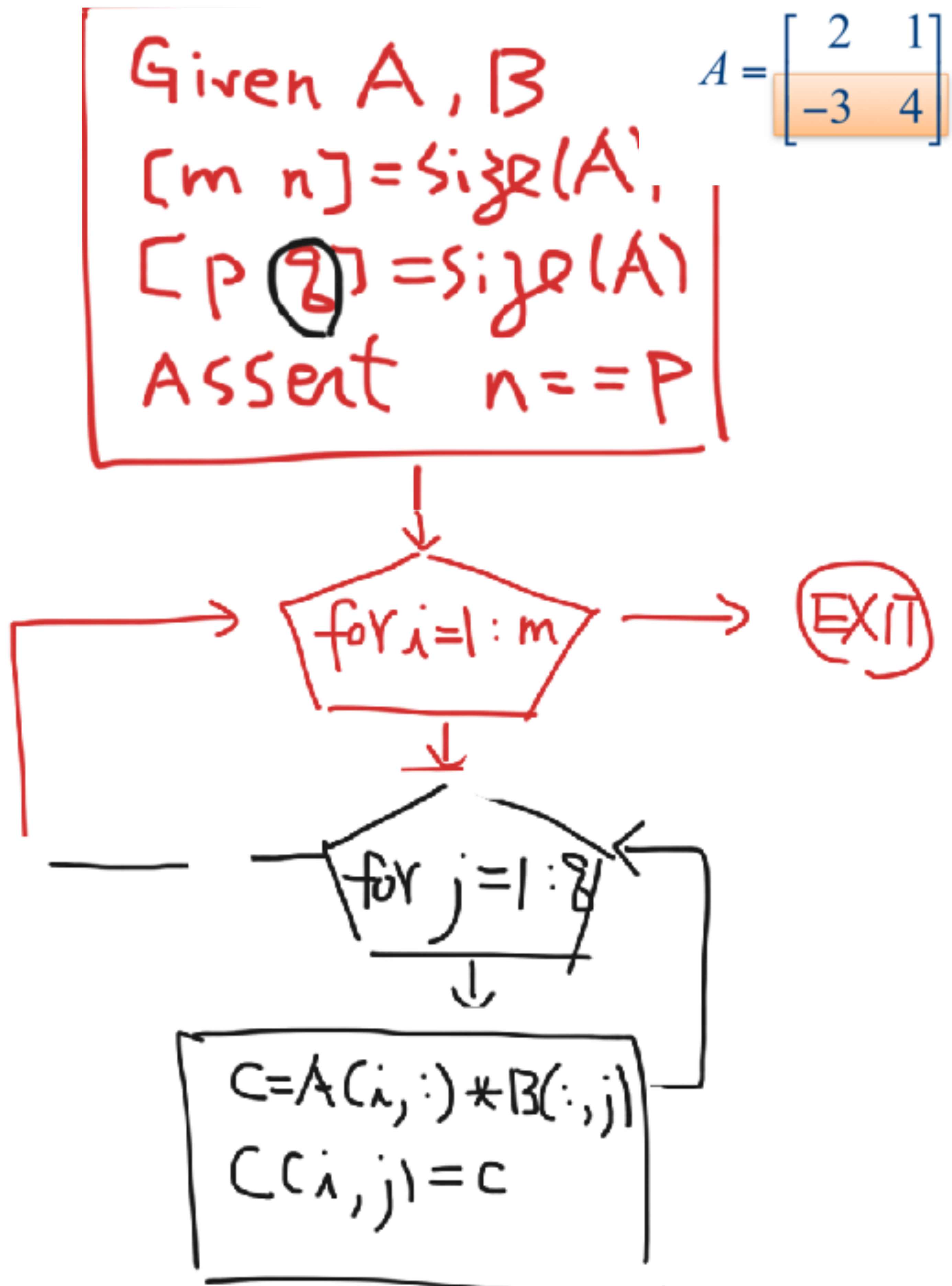
Size of a Product Matrix

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then AB will be an $m \times n$ matrix.

$$\underset{m \times r}{A} \quad \underset{r \times n}{B} \quad = \underset{m \times n}{AB}$$

For example, suppose A is a 5×6 matrix and B is an 6×7 matrix. Matrix A has six columns, whereas B has six rows. Thus AB exits. AB will be a 5×7 matrix.

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -7 & 3 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$



Gauss-Jordan Elimination

Definition

A matrix is in **reduced echelon form** if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each other row is 1. This element is called a **leading 1**.
3. The leading 1 of each after the first is positioned to the right of the leading 1 of the previous row.
4. All other elements in a column that contains a leading 1 are zero.

In Reduced Echelon Form

$$\begin{bmatrix}1 & 0 & 8\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix} \quad \begin{bmatrix}1 & 0 & 0 & 7\\0 & 1 & 0 & 3\\0 & 0 & 1 & 9\end{bmatrix} \quad \begin{bmatrix}1 & 4 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 2 & 3 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 5 & 0 & 0 & 8\\0 & 1 & 7 & 0 & 0 & 9\\0 & 0 & 0 & 1 & 0 & 5\\0 & 0 & 0 & 0 & 1 & 4\end{bmatrix} \quad \begin{bmatrix}1 & 2 & 0 & 3 & 0 & 4\\0 & 0 & 1 & 2 & 0 & 7\\0 & 0 & 0 & 0 & 1 & 6\\0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Not in Reduced Echelon Form

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Row of zeros not at bottom of matrix

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

First nonzero element in row 2 is not 1

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

Leading 1 in row 3 not to the right of leading 1 in row 2

$$\begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Nonzero element above leading 1 in Row 2

Gauss-Jordan Elimination

1. Write down the augmented matrix of the system of linear equations
2. Derive the reduced echelon for of the augmented matrix using elementary row operations. This is done by creating leading 1s, then zeros above and below each leading 1, column by column, starting with the first column.
3. Write down the system of equations corresponding to the reduced echelon form. This system gives the solution.

Example 1

Use the method of Gauss-Jordan elimination to find reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Solution

pivot

$$\underset{R1 \leftrightarrow R2}{\approx} \begin{bmatrix} 3 & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix} \xrightarrow{(1/3)R1} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

$$\xrightarrow{R3+(-\ 4)R1} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \xrightarrow{(1/2)R2} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

pivot

$$\begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \xrightarrow[\text{R3+(- 2)R2}]{\text{R1+R2}} \begin{bmatrix} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & \boxed{1} & -6 \end{bmatrix}$$

pivot

$$\xrightarrow[\text{R2+R3}]{\text{R1+(- 2)R3}} \begin{bmatrix} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

The matrix is the reduced echelon form of the give

```
>> rref(A)

ans =

     1     1     0     0    17
     0     0     1     0    -5
     0     0     0     1    -6
```

Example 2

Solve, if possible, the system of equations

$$\begin{aligned} 3x_1 - 3x_2 + 3x_3 &= 9 \\ 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 - 5x_2 - x_3 &= 7 \end{aligned}$$

Solution

$$\begin{bmatrix} 3 & -3 & 3 & 9 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix} \xrightarrow{(1/3)R1} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix}$$

$$\xrightarrow[R3+(-\ 3)R1]{R2+(-\ 2)R1} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -4 & -2 \end{bmatrix} \xrightarrow[R3 + (2)R2]{R1 + R2} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x_1 + \quad 3x_3 &= 4 \\ x_2 + 2x_3 &= 1 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -3x_3 + 4 \\ x_2 &= -2x_3 + 1 \end{aligned}$$

The general solution to the system is

$$\begin{aligned} x_1 &= -3r + 4 \\ x_2 &= -2r + 1 \\ x_3 &= r \end{aligned}$$

which r is real number (called a parameter).

Example 3

This example illustrates that the general solution can involve a number of parameters. Solve the system of equations

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + 3x_4 &= 4 \\
2x_1 + 4x_2 - 2x_3 + 7x_4 &= 10 \\
-x_1 - 2x_2 + x_3 - 4x_4 &= -6
\end{aligned}
$$

Solution

$$
\begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 7 & 10 \\ -1 & -2 & 1 & -4 & -6 \end{bmatrix}
\xrightarrow[\text{R3+R1}]{\text{R2+(- 2)R1}}
\begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}
$$

$$
\xrightarrow[\text{R3+R2}]{\text{R1+(- 3)R2}}
\begin{bmatrix} 1 & 2 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

We have arrived at the reduced echelon form. The corresponding system of equations is

$$\begin{aligned} x_1 + 2x_2 - x_3 \qquad &= -2 \\ x_4 &= \ \ 2 \end{aligned}$$

Expressing the leading variables in terms of the remaining variables we get

$$x_1 = -2x_2 + x_3 - 2,\ x_4 = 2$$

Let us assign the arbitrary values r to x_2 and s to x_3. The general solution is

$$x_1 = -2r + s - 2,\ x_2 = r,\ x_3 = s,\ x_4 = 2$$

Specific solutions can be obtained by giving r and s various values.

Example 4

This example illustrates a system that has no solution. Let us try to solve the system

$$\begin{aligned} x_1 + \quad x_2 + 5x_3 &= \ \ 3 \\ x_2 + 3x_3 &= -1 \\ x_1 + 2x_2 + 8x_3 &= \ \ 3 \end{aligned}$$

Solution

$$\begin{bmatrix} 1 & 1 & 5 & 3 \\ 0 & 1 & 3 & -1 \\ 1 & 2 & 8 & 3 \end{bmatrix} \xrightarrow{R3+(-\ 1)R1} \begin{bmatrix} 1 & 1 & 5 & 3 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow[R3+(-\ 1)R2]{R1+(-\ 1)R2} \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R2+R3]{R1+(-\ 4)R3} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The last row of this reduced echelon form gives the equation

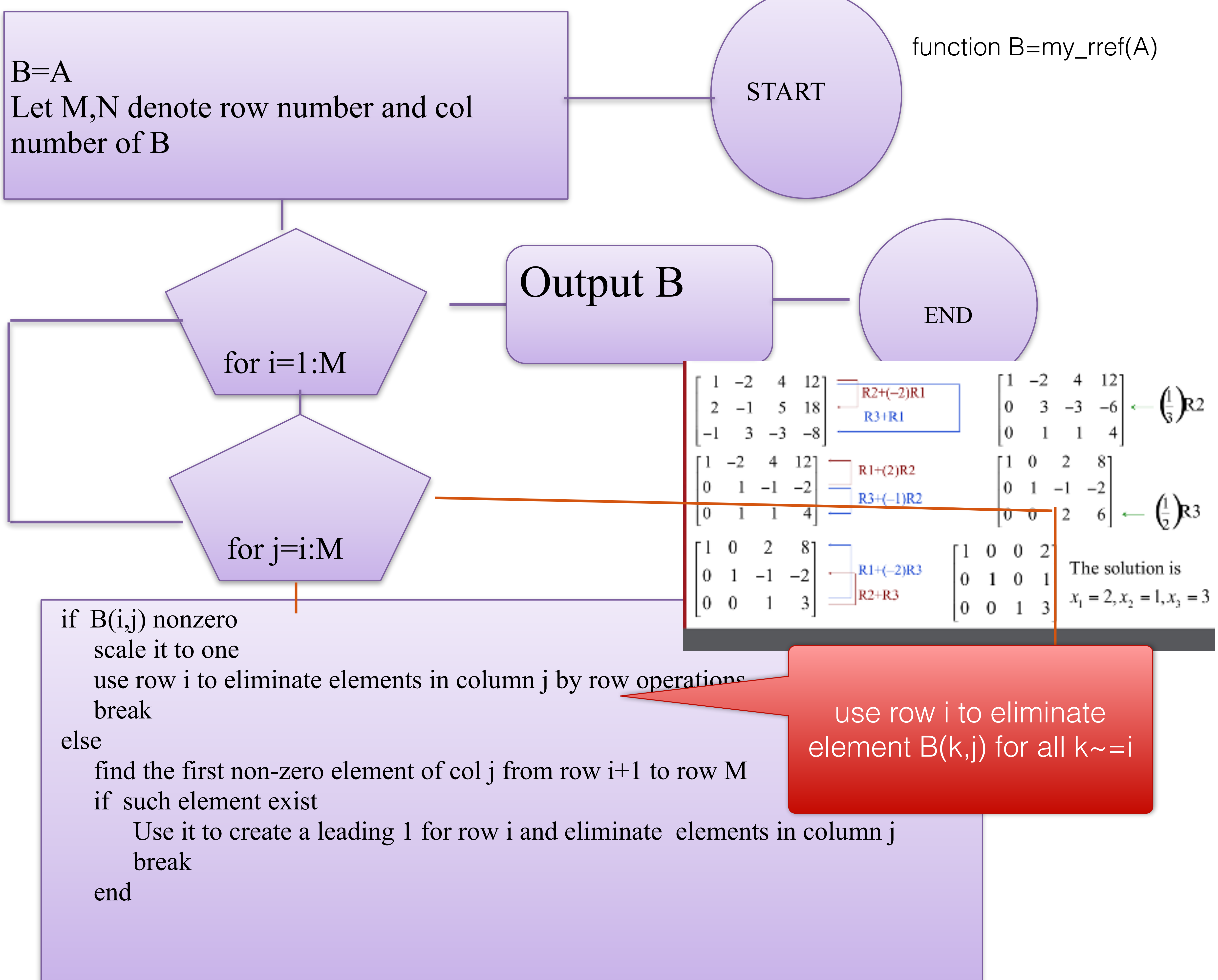
$$0x_1 + 0x_2 + 0x_3 = 1$$

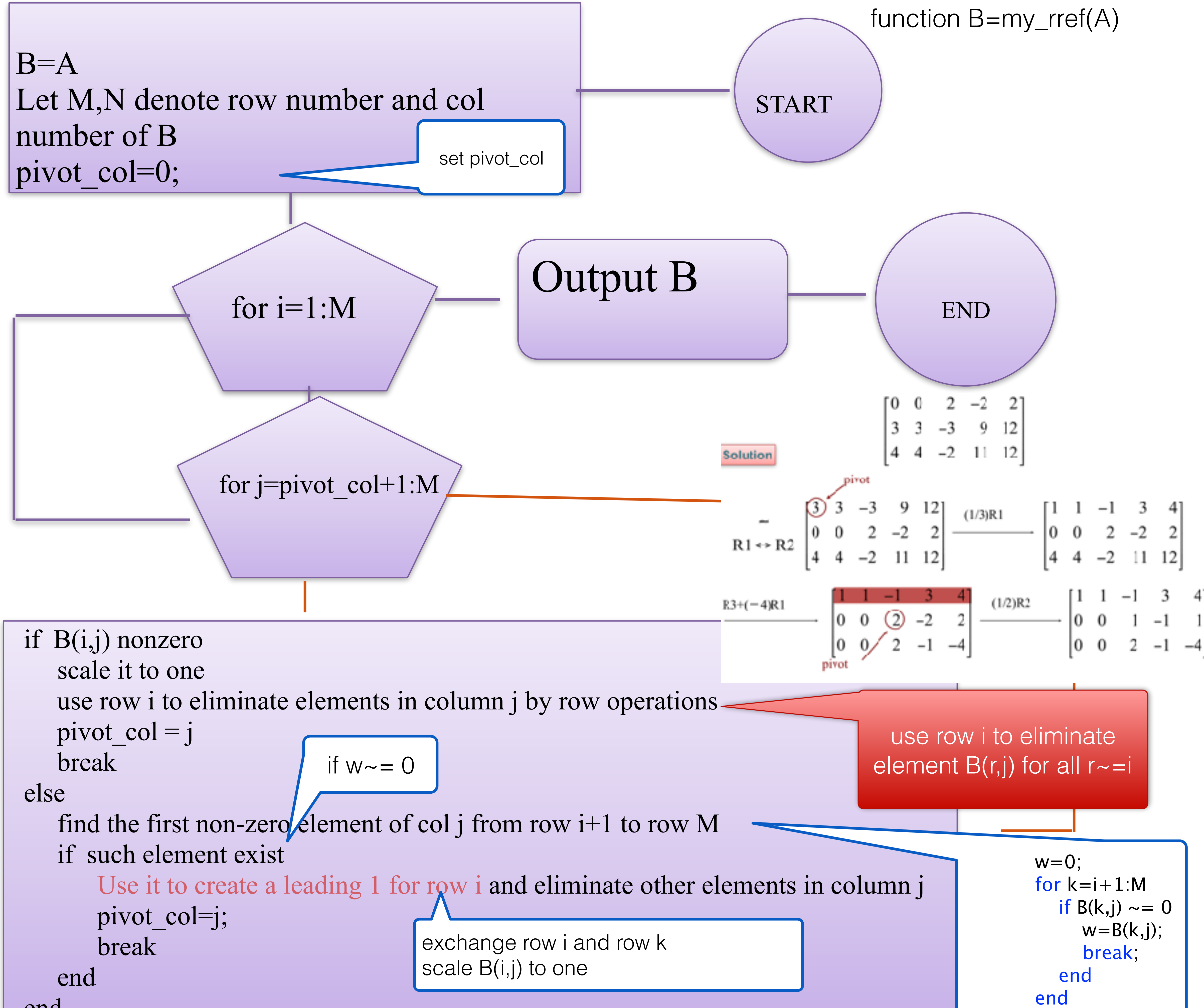
This equation cannot be satisfied for any values of x_1,x_2 and x_3. Thus the system has no solution. (This information was in fact available from the next-to-last matrix.)

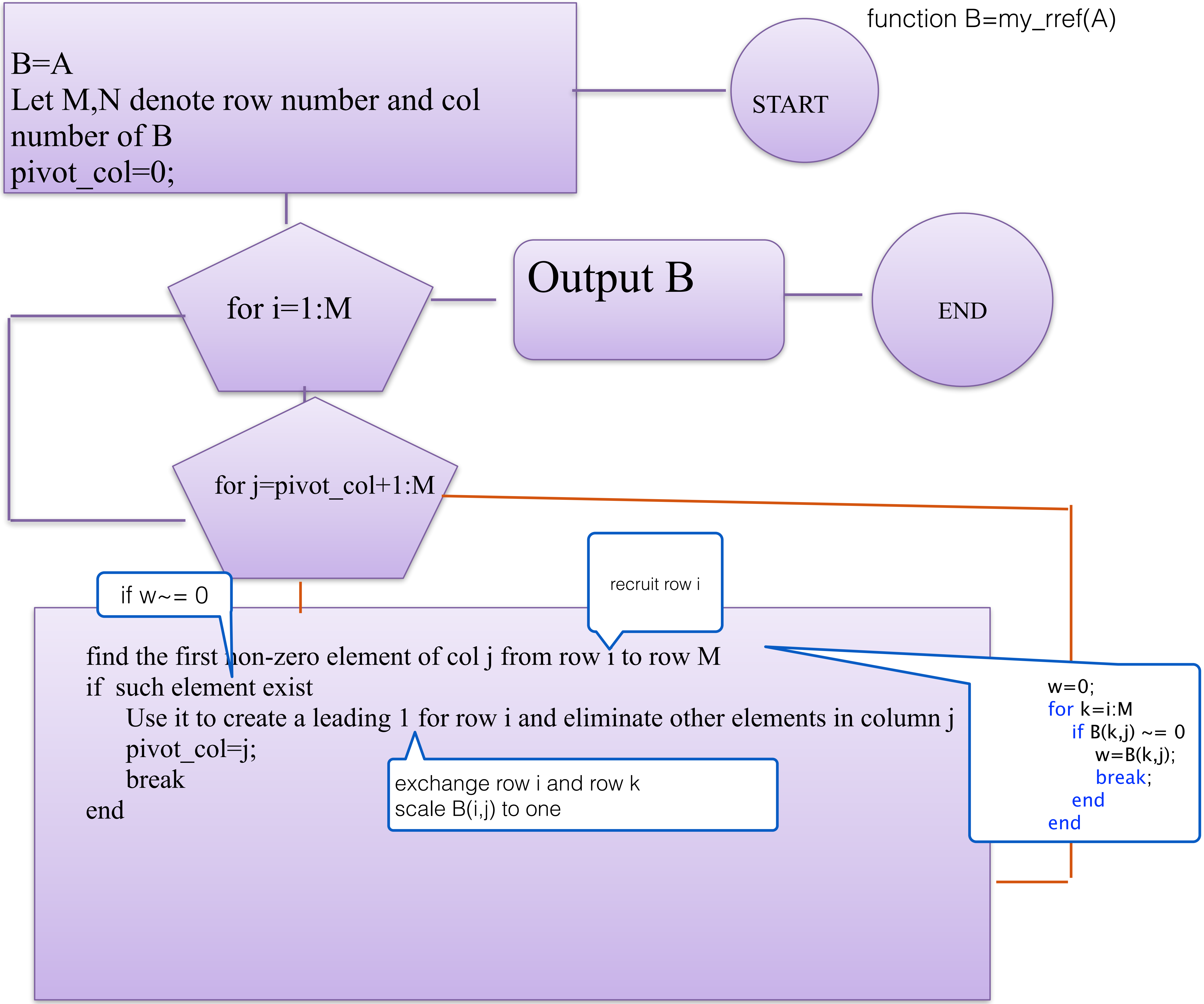
Gauss-Jordan Elimination

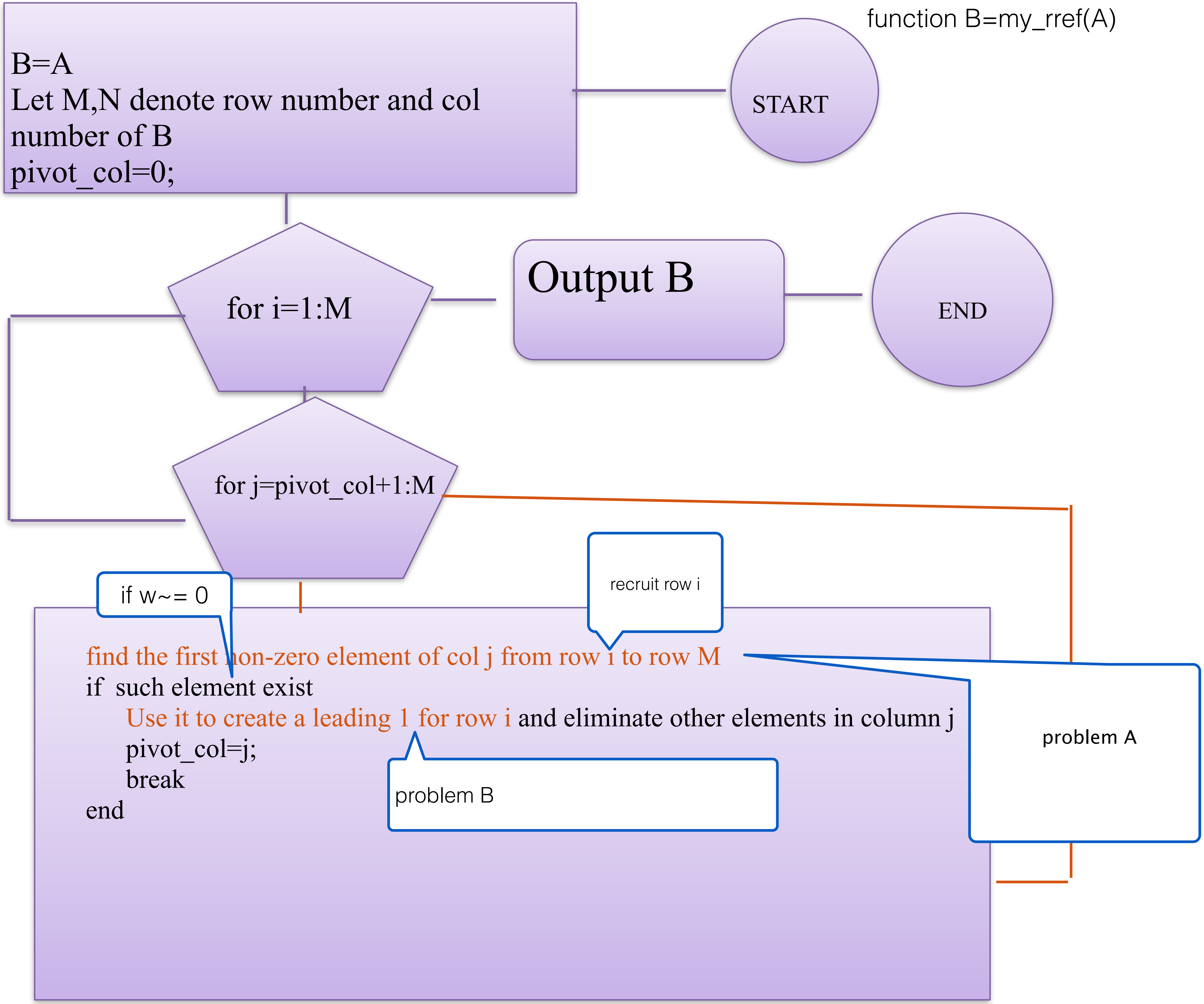
- To solve a system of equations, we can perform **elementary row operations**
  - Interchange two rows of a matrix
  - Multiply the elements of a row by a nonzero constant
  - Add a multiple of the elements of one row to the corresponding elements of another row

function B=my_rref(A)




function B=my_rref(A)
START
B=A
Let M,N denote row number and col number of B
pivot_col=0;
set pivot_col
for i=1:M
Output B
END
for j=pivot_col+1:M
Solution
pivot
R1 <-> R2
(1/3)R1
R3+(−4)R1
(1/2)R2
pivot
if B(i,j) nonzero
scale it to one
use row i to eliminate elements in column j by row operations
pivot_col = j
break
else
find the first non-zero element of col j from row i+1 to row M
if such element exist
Use it to create a leading 1 for row i and eliminate other elements in column j
pivot_col=j;
break
end
end
use row i to eliminate element B(r,j) for all r~=i
if w~= 0
exchange row i and row k
scale B(i,j) to one
w=0;
for k=i+1:M
if B(k,j) ~= 0
w=B(k,j);
break;
end
end


function B=my_rref(A)
START
B=A
Let M,N denote row number and col number of B
pivot_col=0;
for i=1:M
Output B
END
for j=pivot_col+1:M
if w~= 0
recruit row i
find the first non-zero element of col j from row i to row M
if such element exist
Use it to create a leading 1 for row i and eliminate other elements in column j
pivot_col=j;
break
end
exchange row i and row k
scale B(i,j) to one
w=0;
for k=i:M
if B(k,j) ~= 0
w=B(k,j);
break;
end
end


function B=my_rref(A)
START
B=A
Let M,N denote row number and col number of B
pivot_col=0;
for i=1:M
Output B
END
for j=pivot_col+1:M
if w~= 0
recruit row i
find the first non-zero element of col j from row i to row M
if such element exist
Use it to create a leading 1 for row i and eliminate other elements in column j
pivot_col=j;
break
end
problem A
problem B

Determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

Solution

$$[A : I_n] = \left[\begin{array}{cccccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 2 & -3 & -5 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\substack{R2+(-2)R1 \\ R3+R1}}{\approx} \left[\begin{array}{cccccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\underset{(-1)R2}{\approx} \left[\begin{array}{cccccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} \approx \\ R1+R2 \\ R3+(-2)R2 \end{array}\begin{bmatrix} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{c} \approx \\ R1+R3 \\ R2+(-1)R3 \end{array}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

Thus, $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix}$.

```
>> A=[1 -1 -2;2 -3 -5;-1 3 5]; inv(A)

ans =

         0    1.0000    1.0000
    5.0000   -3.0000   -1.0000
   -3.0000    2.0000    1.0000
```

A=[1 -1 -2;2 -3 -5;-1 3 5]; inv(A)
my_rref([A eye(3)])

Determine the inverse of the following matrix, if it exist.

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix}$$

Solution

$$[A : I_3] = \begin{bmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -1 & 4 & 0 & 0 & 1 \end{bmatrix} \underset{\substack{R2+(-1)R1 \\ R3+(-2)R1}}{\approx} \begin{bmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & -3 & -6 & -2 & 0 & 1 \end{bmatrix}$$

$$\underset{\substack{R1+(-1)R2 \\ R3+3R2}}{\approx} \begin{bmatrix} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -5 & 3 & 1 \end{bmatrix}$$

There is no need to proceed further.
The reduced echelon form cannot have a one in the (3, 3) location.
The reduced echelon form cannot be of the form $[I_n : B]$.
Thus A^{-1} does not exist.

```
A=[1 1 5;1 2 7;2 -1 4]; my_rref([A eye(3)])
```

Solving linear systems

- Direct method
  - Analytic approach
  - Naive Gaussian elimination
- Iterative method
  - Jacobi method
  - Gauss-Seidel method
  - SOR method
  - Conjugate gradient method

A linear system

$$\left(\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 12 & -8 & 6 & 10 & 26 \\ 3 & -13 & 9 & 3 & -19 \\ -6 & 4 & 1 & -18 & -34 \end{array}\right)$$

•

$$6x_1 - 2x_2 + 2x_3 + 4x_4 = 16$$
$$12x_1 - 8x_2 + 6x_3 + 10x_4 = 26$$
$$3x_1 - 13x_2 + 9x_3 + 3x_4 = -19$$
$$-6x_1 + 4x_2 + x_3 - 18x_4 = -34$$

Forward elimination

$$\left(\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 12 & -8 & 6 & 10 & 26 \\ 3 & -13 & 9 & 3 & -19 \\ -6 & 4 & 1 & -18 & -34 \end{array}\right)$$

r1*(-2)+r2
r1*(-1/2)+r3
r1+r3

$\Rightarrow$

$$\left(\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & -12 & 8 & 1 & -27 \\ 0 & 2 & 3 & -14 & 18 \end{array}\right)$$

r2*(-3)+r3
r2*(1/2)+r4

$\Downarrow$

$$\left(\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 4 & -13 & -21 \end{array}\right)$$

r3*(-2)+r4

$\Leftarrow$

$$\left(\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right)$$

Triangular linear system

$$\left(\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right)$$

Backward substitution

$$\begin{pmatrix} 6 & -2 & 2 & 4 & | & 16 \\ 0 & -4 & 2 & 2 & | & -6 \\ 0 & 0 & 2 & -5 & | & -9 \\ 0 & 0 & 0 & -3 & | & -3 \end{pmatrix}$$

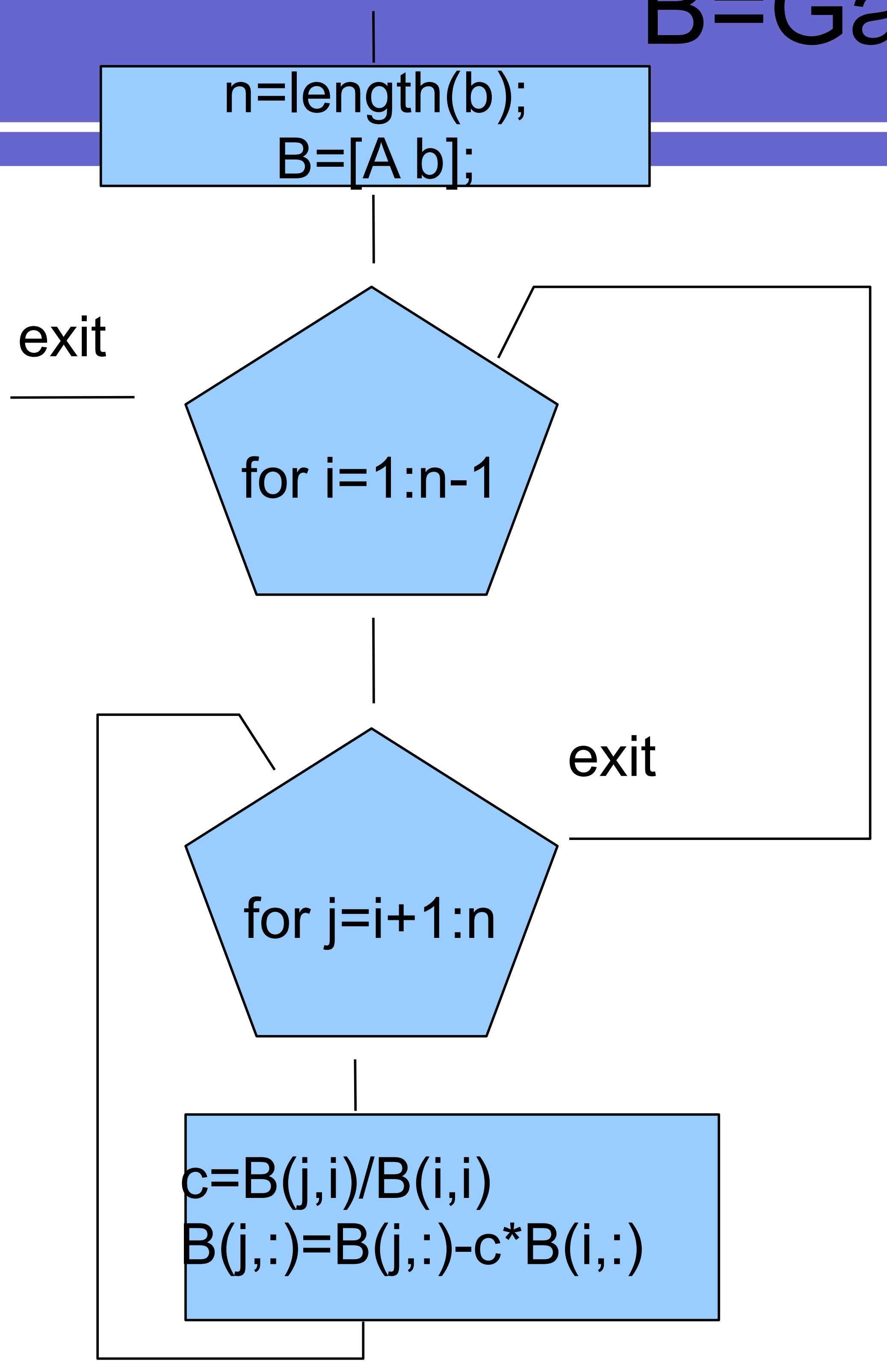
$$6x_1 - 2\times 1 - 2\times 2 + 4\times 1 = 16 \Rightarrow x_1 = 3$$

$$-4x_2 - 2\times 2 + 2\times 1 = -6 \Rightarrow x_2 = 1$$

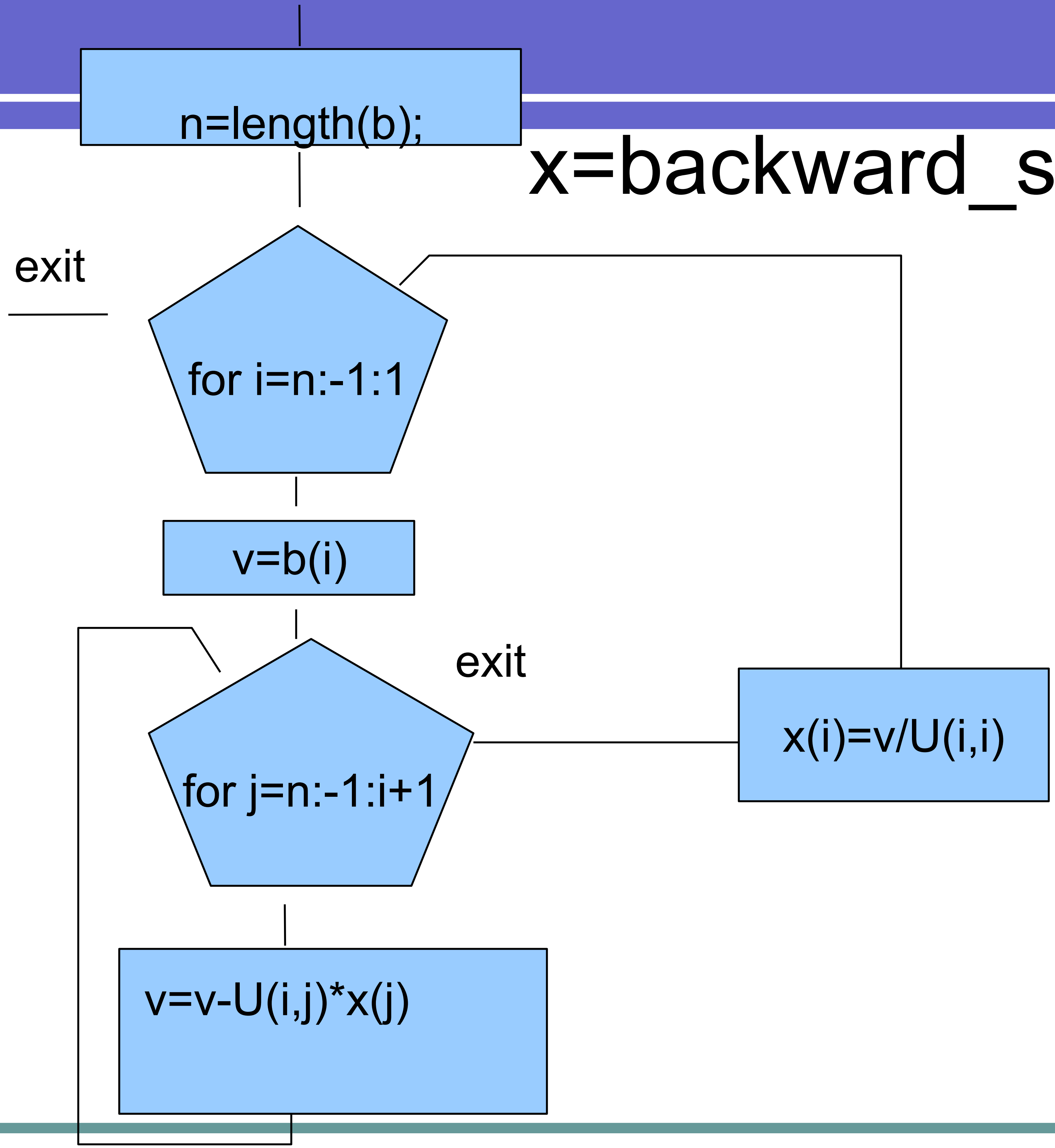
$$2x_3 - 5\times 1 = -9 \Rightarrow x_3 = -2$$

$$x_4 = 1$$

B=Gauss_eli(A,b)



x=backward_sub(U,b)



Exercise

- Implement naive forward elimination and backward substitution for solving a linear system
- Give two examples to test your matlab codes