Linear system solving

An algorithm to find Reduced Echelon Form

Example 1

Use the method of Gauss-Jordan elimination to find reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Solution

$$\underset{R1 \leftrightarrow R2}{\approx} \begin{bmatrix} 3 & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix} \xrightarrow{(1/3)R1} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

pivot

$$\xrightarrow{R3+(-4)R1} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \xrightarrow{(1/2)R2} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

pivot

$$\begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \xrightarrow[R3+(-2)R2]{R1+R2} \begin{bmatrix} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

pivot

$$\xrightarrow[R2+R3]{R1+(-2)R3} \begin{bmatrix} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

The matrix is the reduced echelon form of the given matrix.

Gauss-Jordan Elimination

- To solve a system of equations, we can perform **elementary row operations**
  - Interchange two rows of a matrix
  - Multiply the elements of a row by a nonzero constant
  - Add a multiple of the elements of one row to the corresponding elements of another row

```
>> A = [ 0 0 2 -2 2;3 3 -3 9 12;4 4 -2 11 12]

A =

     0     0     2    -2     2
     3     3    -3     9    12
     4     4    -2    11    12
```

Row operation I

Multiply elements of a row by a non-zero constant

function A = row_op1(A, i, s)

```
% multiply s to row i
```

Exercise 1
Write a Matlab function for row_op1

```
>> A = [ 0 0 2 -2 2;3 3 -3 9 12;4 4 -2 11 12]

A =

     0     0     2    -2     2
     3     3    -3     9    12
     4     4    -2    11    12

>> A = row_op1(A, 2, 2)

A =

     0     0     2    -2     2
     6     6    -6    18    24
     4     4    -2    11    12
```

Exercise 2
Give an
example to
test row_op1

Row operation II
Swap row i and row j

```
function  A = row_op2(A, i, j)
% swap row i and row j
```

Exercise 3
Write a Matlab function for row_op2

```
>> A = [ 0 0 2 -2 2;3 3 -3 9 12;4 4 -2 11 12]

A =

     0     0     2    -2     2
     3     3    -3     9    12
     4     4    -2    11    12
```

Exercise 4
Give an example to test row_op2

```
>> A = row_op2(A,1,2)

A =

     3     3    -3     9    12
     0     0     2    -2     2
     4     4    -2    11    12
```

Row operation III

Add a multiple of row j to row i

```
function  A = row_op3(A, i, j, s)
% add a multiple of row j to
% row i
```

Exercise 5
Write a Matlab function for row_op3

```
>> A = [ 0 0 2 -2 2;3 3 -3 9 12;4 4 -2 11 12]

A =

     0     0     2    -2     2
     3     3    -3     9    12
     4     4    -2    11    12

>> A = row_op3(A,2, 3, 2)

A =

     0     0     2    -2     2
    11    11    -7    31    36
     4     4    -2    11    12
```

Exercise 6
Give an example to test row_op3

Example 1

Use the method of Gauss-Jordan elimination to find reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

A = row_op2(A,1,2);
A= row_op1(A,1,1/3);
A=row_op3(A, 3,1, -4);
A=row_op1(A,2,1/2)

Solution

A = row_op2(A,1,2);

pivot

A= row_op1(A,1,1/3);

$$\underset{R1 \leftrightarrow R2}{\approx} \begin{bmatrix} 3 & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix} \xrightarrow{(1/3)R1} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

$$\xrightarrow{R3+(-4)R1} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \xrightarrow{(1/2)R2} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

A=row_op3(A, 3,1, -4);

pivot

A=row_op1(A,2,1/2)

```
>> A = [ 0 0 2 -2 2;3 3 -3 9 12;4 4 -2 11 12]

A =

     0     0     2    -2     2
     3     3    -3     9    12
     4     4    -2    11    12

>> A = row_op2(A,1,2);
A= row_op1(A,1,1/3);
A=row_op3(A, 3,1, -4);
A=row_op1(A,2,1/2)

A =

     1     1    -1     3     4
     0     0     1    -1     1
     0     0     2    -1    -4
```

Exercise 7
Give row operations to find rref of matrix A

A = row_op3(A,1,2,1);
A=row_op3(A,3,2,-2);
A=row_op3(A,1,3,-2);
A=row_op3(A,2,3,1)

A = row_op3(A,1,2,1);

$$\begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \xrightarrow[R3+(-2)R2]{R1+R2} \begin{bmatrix} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

pivot

A=row_op3(A,3,2,-2);

A=row_op3(A,1,3,-2);

$$\xrightarrow[R2+R3]{R1+(-2)R3} \begin{bmatrix} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

A=row_op3(A,2,3,1)

The matrix is the reduced echelon form of the given matrix.

```
>> A = row_op3(A,1,2,1);
A=row_op3(A,3,2,-2);
A=row_op3(A,1,3,-2);
A=row_op3(A,2,3,1)

A =

     1     1     0     0    17
     0     0     1     0    -5
     0     0     0     1    -6
```

Exercise 7 Give row operations to find rref of matrix A

```
function B=my_rref(A)
```

```
B=A;
[M N]=size(B);
pivot_col = 0;
```

```
for i = 1:M
    for j=pivot_col+1:N

        find the first non-zero element of col j from row i to row M

        if such element exist

            use it to create a leading 1 for row i

            eliminate other elements in column j

            pivot_col=j;

            break

        end

    end
end
```

Exercise 8
Write a Matlab function to implement my_rref

find the first non-zero element of col j from row i to row M

```
>> A = [ 0 0 2 -2 2;3 3 -3 9 12;4 4 -2 11 12]

            j:2
A =

      0     0     2    -2     2
i:2   3     3    -3     9    12
      4     4    -2    11    12

>> i = 2; M = 3; j = 2;
>> ind = find(A(i:M,j)~=0);
>> length(ind)

ans =

     2
```

i: 2
M: 3
j: 2
Such element does not exist

find the first non-zero element of col j from row i to row M

```
>> A = [ 1 2 0 0 1;0 0 0 1 1; 0 0 0 1 0; 0 0 1 1 1]

               j:2
A =

          1    2    0    0    1
i:2       0    0    0    1    1
          0    0    0    1    0
          0    0    1    1    1

>> ind = find(A(2:4,2)~=0);
>> length(ind)

ans =

     0
```

i: 2
M: 4
j: 2
Such element does not exist

find the first non-zero element of col j from row i to row M

```
>> A = [ 1 2 0 0 1;0 0 0 1 1; 0 0 0 1 0; 0 0 1 1 1]
              j:3
A =

             1    2    0    0    1
i:2   -->    0    0    0    1    1
             0    0    0    1    0
             0    0    1    1    1

>> i = 2; M = 4; j = 3;
>> ind = find(A(i:M,j)~=0);
>> length(ind)

ans =

     1

>> A(i-1+ind(1),j)

ans =

     1
```

Such element exists

The first non-zero element

How to use such element to create a leading in row i?

Apply row_op2 to swap row i-1+ind(1) and row i
Apply row_op1 to create a leading 1

```
>> B = my_rref(A)

B =

     1     1     0     0    17
     0     0     1     0    -5
     0     0     0     1    -6
```

Exercise 9 Give two examples to test my_rref

Determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

Solution

$$[A : I_n] = \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 2 & -3 & -5 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\substack{\mathrm{R2}+(-2)\mathrm{R1} \\ \mathrm{R3}+\mathrm{R1}}}{\approx} \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix}$$

$$\underset{(-1)\mathrm{R2}}{\approx} \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix}$$

```
>> A=[1 -1 -2;2 -3 -5;-1 3 5]; my_rref([A eye(3)])

ans =

     1     0     0     0     1     1
     0     1     0     5    -3    -1
     0     0     1    -3     2     1
```

Exercise 10
Find inervse of A using my_rref

```
>> inv(A)

ans =

         0    1.0000    1.0000
    5.0000   -3.0000   -1.0000
   -3.0000    2.0000    1.0000
```