Conjugate gradient Method

General Algorithm

Initialize x_0 , $r_0 = \mathrm{b} - \mathrm{A}x_0$, $p_0 = r_0$

repeat

$$\alpha_k = \frac{r_k^T r_k}{p_k^T A p_k} \qquad (Lemma.3)$$

$$x_{k+1} = x_k + \alpha_k p_k$$

$$r_{k+1} = r_k - \alpha_k A p_k \quad (r_{k+1} = \mathrm{b} - \mathrm{A}x_{k+1})$$

if r_{k+1} is sufficiently small **then** exit loop **end if**

$$\beta_k = \frac{r_{k+1}^T r_{k+1}}{r_k^T r_k} \qquad (Lemma.3)$$

$$p_{k+1} = r_{k+1} + \beta_k p_k$$

until stopping condition is satisfied

$$\alpha_k = \frac{p_k^T r_k}{p_k^T A p_k}$$

r_k : negative gradient

$$\because \; p_k = r_k + \beta_{k-1} p_{k-1}$$

r_k and p_{k-1} are conjugate

$$\Rightarrow \quad \alpha_k = \frac{r_k^T r_k}{p_k^T A p_k}$$

The β is chosen such that p_{k+1} is conjugate to p_k

$$\because \; p_{k+1} = r_{k+1} + \beta_k p_k$$

$$\beta_k = - \frac{r_{k+1}^T A p_k}{p_k^T A p_k}$$

(1) $r_k = r_{k-1} - \alpha_{k-1} A p_{k-1}$

$$A p_{k-1} = \frac{1}{\alpha_{k-1}} (r_{k-1} - r_k)$$

(2) The numerator

$$r_{k+1}^T A p_k = \frac{1}{\alpha_k} r_{k+1}^T (r_k - r_{k+1})$$

$$= -\frac{1}{\alpha_k} r_{k+1}^T r_{k+1}$$

The denominator

$$p_k^T A p_k = (r_k^T + \beta_{k-1} p_{k-1}^T)$$

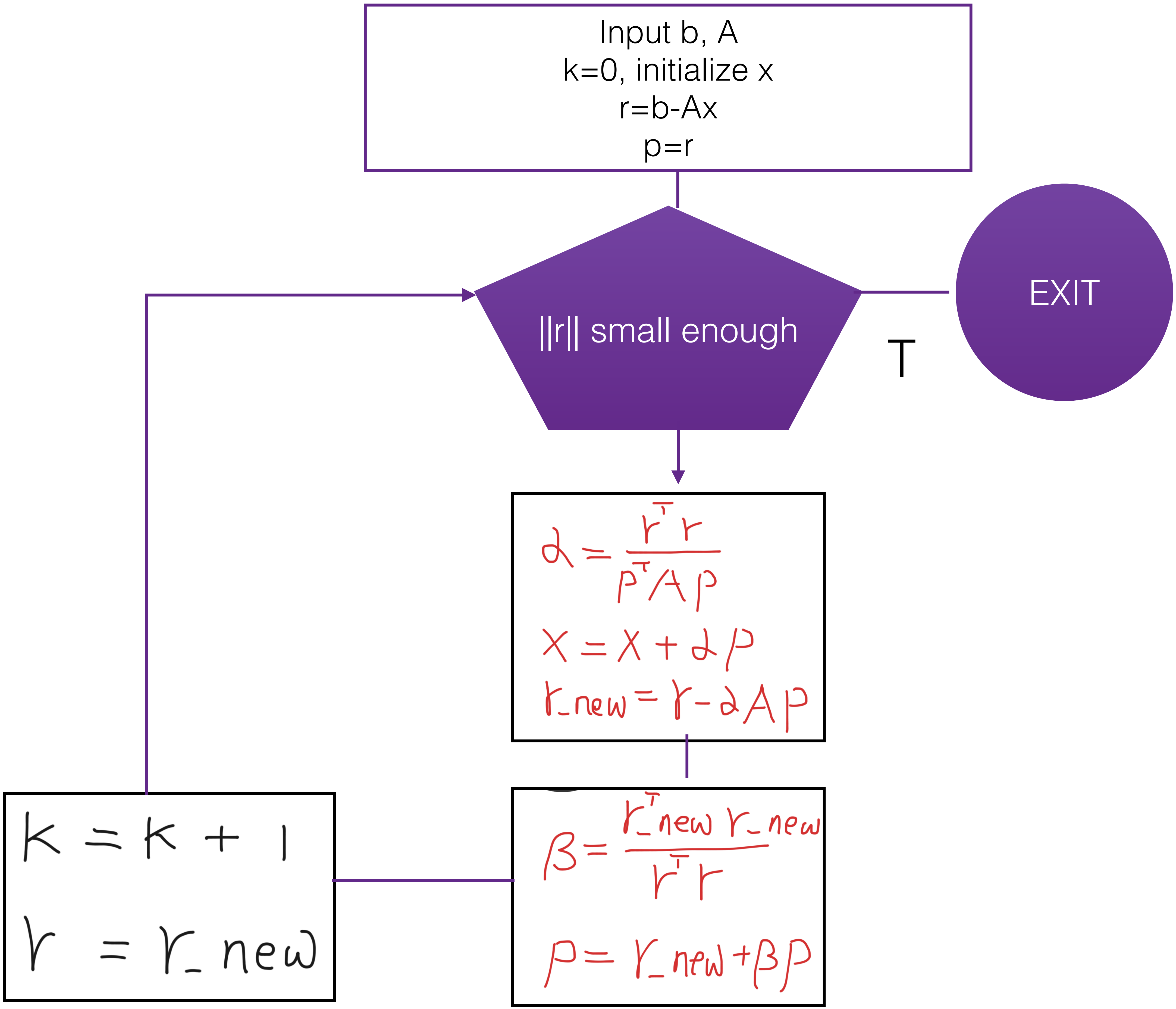
$$A p_k$$

$$= r_k^T A p_k = \frac{1}{\alpha_k} r_k^T (r_k - r_{k-1})$$

$$= \frac{1}{\alpha_k} r_k^T r_k$$

$$\beta_k = -\frac{r_{k+1}^T A p_k}{p_k^T A p_k}$$

$$\beta_k = \frac{r_{k+1}^T r_{k+1}}{r_k^T r_k}$$


Input b, A
k=0, initialize x
r=b-Ax
p=r
||r|| small enough
T
EXIT
$\alpha = \frac{r^T r}{p^T A p}$
$x = x + \alpha p$
$r\_new = r - \alpha A p$
$\beta = \frac{r\_new^T \, r\_new}{r^T r}$
$p = r\_new + \beta p$
$k = k + 1$
$r = r\_new$