A system of nonlinear equations

Newton's method

Outline

- Matlab toolbox
- Newton's method for nonlinear system solving
  - Updating rule
  - Matlab implementation

```
function F = myfun(x)
   F(1) =  x(1)^2 + x(2)^2-1;
   F(2) =  x(1)^2 - x(2)^2;
return
```

$$f_1(x_1, x_2) = x_1^2 + x_2^2 - 1$$

$$f_2(x_1, x_2) = x_1^2 - x_2^2$$

symbols

```
s1='x1^2+x2^2-1';
s2='x1^2-x2^2';
x1=sym('x1')
x2=sym('x2')
```

Inline Function

```
f=inline([sym(s1);sym(s2)]);
f(0,0)
```

fsolve

```
x=fsolve(@(x) [x(1)^2+x(2)^2-1
  x(1)^2-x(2)^2],[1 1])
```

```
x =

    0.7071    0.7071
```

```
s1='x1^2+x2^2-1';
s2='x1^2-x2^2';

x1=sym('x1')
x2=sym('x2')
f=inline([sym(s1);sym(s2)]);
f(x(1),x(2))
```

```
ans =

   1.0e-11 *

    0.2282
         0
```

zeros

Jacobian

```
A=jacobian([sym(s1);sym(s2)],[x1 x2]);
j=inline(A);
j(1,1)
```

$$f_1(x_1, x_2) = x_1^2 + x_2^2 - 1$$

$$f_2(x_1, x_2) = x_1^2 - x_2^2$$

```
A =

[ 2*x1,  2*x2]
[ 2*x1, -2*x2]
```

fsolve

https://www.mathworks.com/help/optim/ug/fsolve.htm
mathwork

$$e^{-e^{-(x_1+x_2)}} - x_2\left(1+x_1^2\right) = 0$$
$$x_1\cos\left(x_2\right) + x_2\sin\left(x_1\right) - \tfrac{1}{2} = 0.$$

Write a function that computes the left-hand side of these two equations.

```
function F = root2d(x)

F(1) = exp(-exp(-(x(1)+x(2)))) - x(2)*(1+x(1)^2);
F(2) = x(1)*cos(x(2)) + x(2)*sin(x(1)) - 0.5;
```

[1] Coleman, T.F. and Y. Li, "An Interior, Trust Region Approach for Nonlinear Minimization Subject to Bounds," *SIAM Journal on Optimization*, Vol. 6, pp. 418-445, 1996.

[2] Coleman, T.F. and Y. Li, "On the Convergence of Reflective Newton Methods for Large-Scale Nonlinear Minimization Subject to Bounds," *Mathematical Programming*, Vol. 67, Number 2, pp. 189-224, 1994.

[3] Dennis, J. E. Jr., "Nonlinear Least-Squares," *State of the Art in Numerical Analysis*, ed. D. Jacobs, Academic Press, pp. 269-312.

[4] Levenberg, K., "A Method for the Solution of Certain Problems in Least-Squares," *Quarterly Applied Mathematics 2*, pp. 164-168, 1944.

[5] Marquardt, D., "An Algorithm for Least-squares Estimation of Nonlinear Parameters," *SIAM Journal Applied Mathematics*, Vol. 11, pp. 431-441, 1963.

[6] Moré, J. J., "The Levenberg-Marquardt Algorithm: Implementation and Theory," *Numerical Analysis*, ed. G. A. Watson, Lecture Notes in Mathematics 630, Springer Verlag, pp. 105-116, 1977.

[7] Moré, J. J., B. S. Garbow, and K. E. Hillstrom, *User Guide for MINPACK 1*, Argonne National Laboratory, Rept. ANL-80-74, 1980.

[8] Powell, M. J. D., "A Fortran Subroutine for Solving Systems of Nonlinear Algebraic Equations," *Numerical Methods for Nonlinear Algebraic Equations*, P. Rabinowitz, ed., Ch.7, 1970.

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A method for the solution of certain non-linear problems in least squares

An Algorithm for Least-Squares Estimation of Nonlinear Parameters

Donald W. Marquardt

Journal of the Society for Industrial and Applied Mathematics, Vol. 11, No. 2 (Jun., 1963), 431-441.

Stable URL:
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Nonlinear systems

A system of nonlinear equations

$$F(x_1, x_2, \ldots, x_n) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

$f_1, f_2, \ldots, f_n$ are coordinate functions of F

Example

$$3x_1 - \cos(x_2 x_3) - \frac{1}{2} = 0$$

$$x_1^2 - 81(x_2 + 0.1)^2 + \sin(x_3) + 1.06 = 0$$

$$e^{-x_1 x_2} + 20x_3 + \frac{1}{3}(10\pi - 3) = 0$$

myfun

```
function F = myfun(x)
    F(1) =  3*x(1)-cos(x(2)*x(3))-1/2;
    F(2) =  x(1).^2 -81*(x(2)+0.1).^2+sin(x(3))+1.06;
    F(3) = exp(-x(1)*x(2))+20*x(3)+1/3*(10*pi-3);
return
```

Example

$$x_1^2 + x_2^2 + x_3^2 = 4$$

$$2x_1 - x_2 + x_3 = 1$$

$$x_1 + 3x_2 - x_3 = 3$$

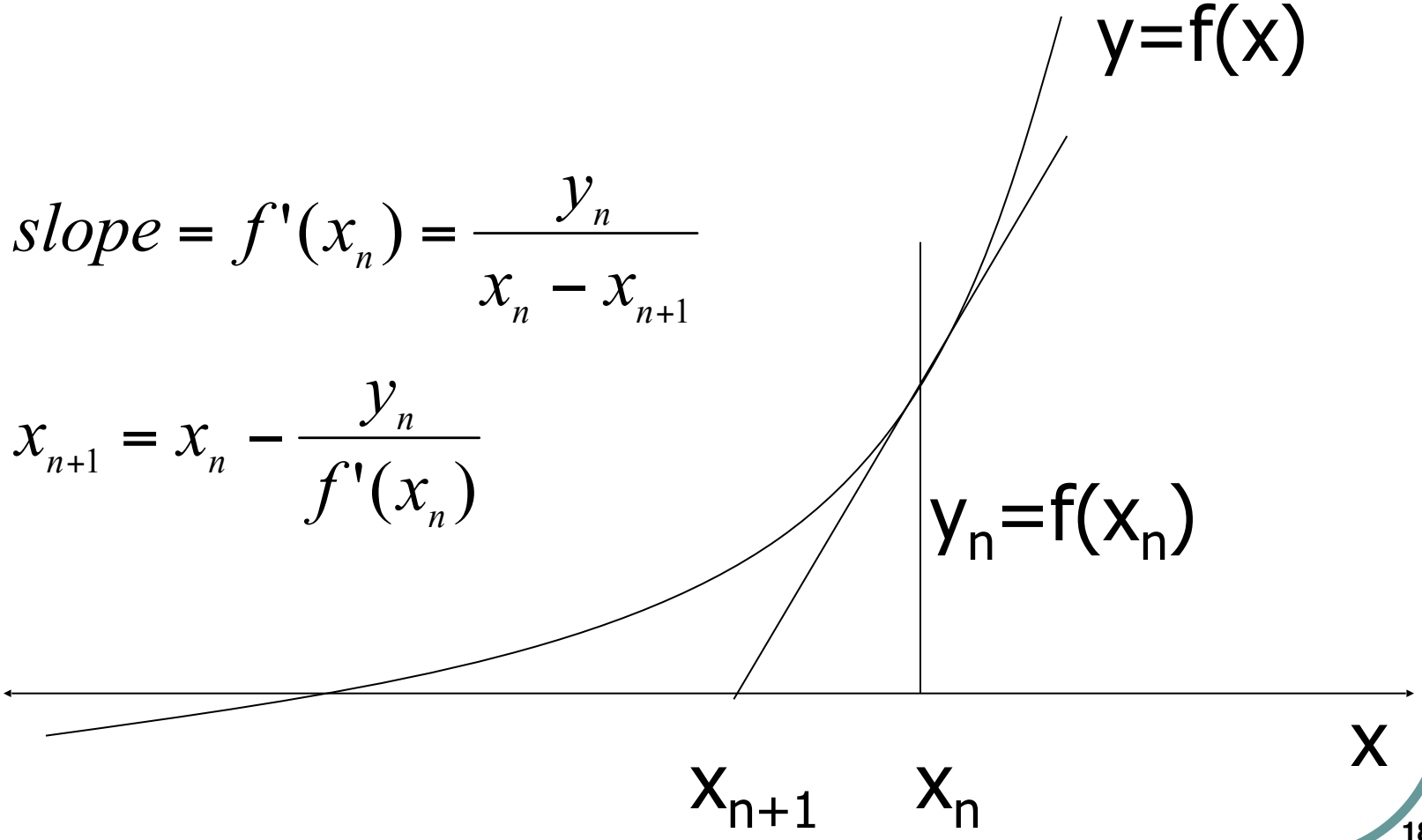
Neural dynamics

$$x_i = \tanh(a_i^T \mathbf{x}), \text{ for all } i$$

Newton's method -Tangent line

$$slope = f'(x_n) = \frac{y_n}{x_n - x_{n+1}}$$

$$x_{n+1} = x_n - \frac{y_n}{f'(x_n)}$$



Updating rule

x : scalar

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

x : vector

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \left[J(\mathbf{x}_n)\right]^{-1} F(\mathbf{x}_n)$$

Taylor series

- Second order expansion at x= x_n

$$F(\mathbf{x}+\Delta\mathbf{x}) \approx F(\mathbf{x}) + J(\mathbf{x})\Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T H(\mathbf{x})\Delta\mathbf{x}$$

Jacobi matrix

Hessian matrix

$$\mathbf{x} \leftarrow \mathbf{x}_n, \quad \Delta\mathbf{x} \leftarrow \mathbf{x} - \mathbf{x}_n,$$

$$F(\mathbf{x}) \approx F(\mathbf{x}_n) + J(\mathbf{x}_n)(\mathbf{x}-\mathbf{x}_n) + \frac{1}{2}(\mathbf{x}-\mathbf{x}_n)^T H(\mathbf{x}_n)(\mathbf{x}-\mathbf{x}_n)$$

Hessian Matrix

$$H(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f_1(\mathbf{x})}{\partial x_1^2} & \frac{\partial^2 f_1(\mathbf{x})}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f_1(\mathbf{x})}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f_2(\mathbf{x})}{\partial x_1 \partial x_2} & \frac{\partial^2 f_2(\mathbf{x})}{\partial x_2^2} & \cdots & \frac{\partial^2 f_2(\mathbf{x})}{\partial x_1 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f_n(\mathbf{x})}{\partial x_n \partial x_1} & \frac{\partial^2 f_n(\mathbf{x})}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f_n(\mathbf{x})}{\partial x_n^2} \end{bmatrix}$$

Jacobi matrix

$$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x})}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n(\mathbf{x})}{\partial x_1} & \frac{\partial f_n(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

First order expansion

- *Set* zero to

$$F(\mathbf{x}) \approx F(\mathbf{x}_n) + J(\mathbf{x}_n)(\mathbf{x} - \mathbf{x}_n)$$

$$F(\mathbf{x}_n) + J(\mathbf{x}_n)(\mathbf{x} - \mathbf{x}_n) = 0 \;\Rightarrow\; \mathbf{x} = \mathbf{x}_n - J^{-1}(\mathbf{x}_n)F(\mathbf{x}_n)$$

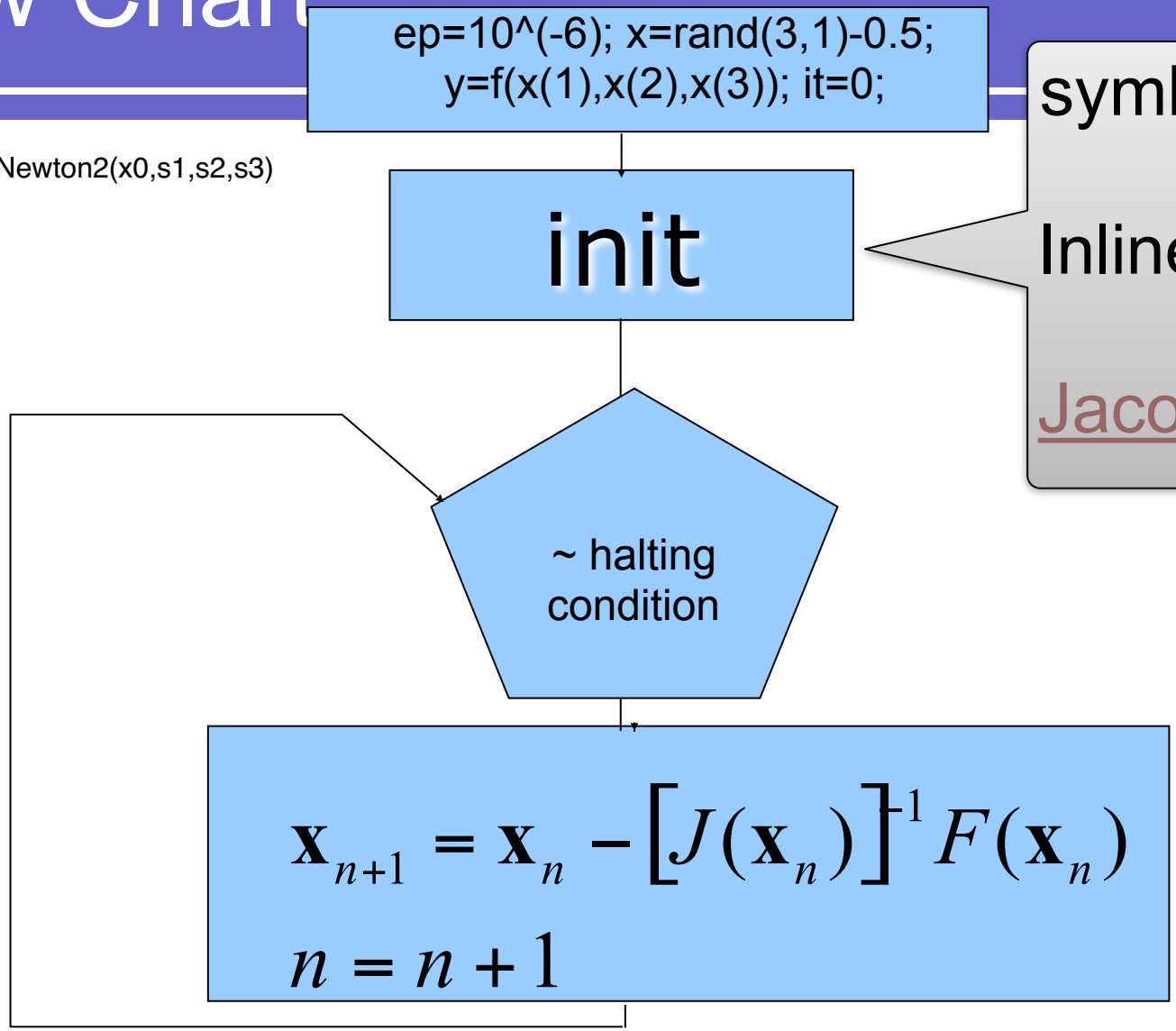
Newton's method

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \left[J(\mathbf{x}_n)\right]^{-1} F(\mathbf{x}_n)$$

$$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x})}{\partial x_n} \\ \vdots & \vdots & & \text{M} \\ \frac{\partial f_n(\mathbf{x})}{\partial x_1} & \frac{\partial f_n(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

Flow Chart

function x=Newton2(x0,s1,s2,s3)



symbols

Inline function

Jacobian

```
s1='3*x1-cos(x2*x3)-1/2';
s2='x1^2 -81*(x2+0.1)^2+sin(x3)+1.06';
s3='exp(-x1*x2)+20*x3+1/3*(10*pi-3)';
x1=sym('x1')
x2=sym('x2')
X3=sym('x3')
```

inline Function

```
f=inline([sym(s1);sym(s2) ;sym(s3)]);
f(0,0,0)
```

Jaconian

```
A=jacobian([sym(s1);sym(s2) ;sym(s3)],[x1
   x2 x3]);
j=inline(A);
j(1,1,1)
```

Symbols, inline and Jacobian

```
s1='3*x1-cos(x2*x3)-1/2';
s2='x1^2 -81*(x2+0.1)^2+sin(x3)+1.06';
s3='exp(-x1*x2)+20*x3+1/3*(10*pi-3)';
x1=sym('x1');x2=sym('x2');x3=sym('x3');
f=inline([sym(s1);sym(s2) ;sym(s3)]);
A=jacobian([sym(s1);sym(s2) ;sym(s3)],[x1 x2 x3]);
j=inline(A);
```

symbols

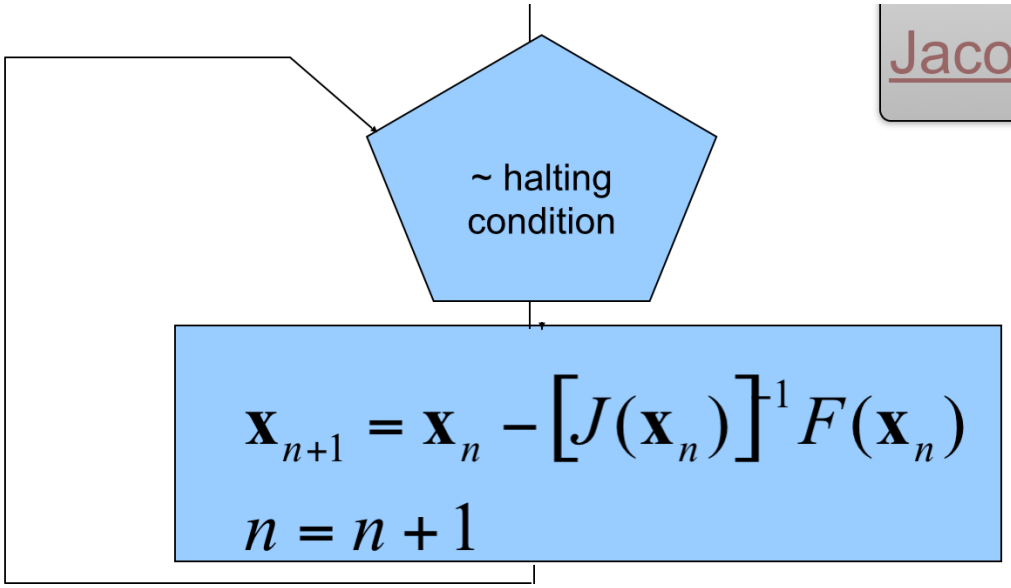
Inline function

Jacobian

Init

```
ep=10^(-6); x=rand(3,1)-0.5;
y=f(x(1),x(2),x(3)); it=0;
```

```
while sum(abs(y)) > ep & it < 100
x=x-inv(j(x(1),x(2),x(3)))*y;
y=f(x(1),x(2),x(3))
it=it+1
end
x
```



Jacob

- Implement the Newton's method for solving a three-variable nonlinear system
- Test your matlab codes with the following nonlinear system

$$3x_1 - \cos(x_2 x_3) - \frac{1}{2} = 0$$

$$x_1^2 - 81(x_2 + 0.1)^2 + \sin(x_3) + 1.06 = 0$$

$$e^{-x_1 x_2} + 20x_3 + \frac{1}{3}(10\pi - 3) = 0$$

$$x_1^2 + x_2^2 + x_3^2 = 4$$

$$2x_1 - x_2 + x_3 = 1$$

$$x_1 + 3x_2 - x_3 = 3$$

```
load mnist_uint8;
train_x = double(train_x) / 255;
test_x  = double(test_x)  / 255;
train_y = double(train_y);
test_y  = double(test_y);
X=[];
for i=1:10
A = train_x;
b = train_y(:,i)==1;
AA=A'*A;
bb=A'*b;
x=CG_lin(AA,bb);
b_hat=A*x>0.5;
er_train=b-b_hat;
b_test=(test_x*x)>0.5;
er_test=b_test-test_y(:,i)==1;
fprintf('%d er_train %d er_test %d\n',i-1,sum(abs(er_train))/length(er_train),sum(abs(er_test))/
length(er_test));
X=[X x];
end
train_y_hat=train_x*X;
r=count_er(train_y,train_y_hat)

test_y_hat=test_x*X;
r=count_er(test_y,test_y_hat)
```

```
function r=count_er(y,y_hat)
[v ind_y]=max(y');
[v ind_y_hat]=max(y_hat');
r=sum(ind_y~=ind_y_hat)/length(ind_y);
```

Let y=f(x) denote a mapping realized by a deep neural network

$$f(x) = W_3 * tanh(W_2 * tanh(W_1 x))$$

where W_1, W_2 and W_3 denote matrixes, x denotes a stimulus vector and y denotes an output vector. For example, x is a handwritten digit and y is a unit vector for representing a label. Consider training and testing sets of MNIST. Discuss how to train W_1, W_2 and W_3 by the Newton method.