Mathematical Software Design
Final problem set

June 17, 2010

1. Let n be a decimal number and $s = [s_1, ..., s_k]$ denote its octal representation, where elements in s belong $\{0, 1, ..., 7\}$. It follows

$$\begin{aligned} n &= \sum_{i=1}^{k} s_i 8^{k-i} \\ &= s_1 8^{k-1} + s_2 8^{k-2} + ... + s_{k-1} 8 + s_k 8^0. \end{aligned}$$

   (a) Draw a flow chart to translate n to s.

   (b) Write a MATLAB function to implement the chart.

   (c) Write a MATLAB function to translate s to n.

2. * Let C denote a 4-by-4 matrix with elements belonging $\{'1', '2', '3', '4'\} \cup \{b\}$, where b denotes a blank.

   (a) Draw a flow chart to generate a matrix C that satifies the following conditions.

      1. C contain only 4 non-blank elements.
      2. All non-blank elements in C are different.
      3. There is exactly only one non-blank element in each row and each column of C.

   (b) Write a matlab function to implement the flow chart.

3. * Let C denote a 4-by-4 matrix with elements belonging $\{'1', '2', '3', '4'\}$. A row in C is valid if it exactly contains four different elements.

   (a) Draw a flow chart to count the number of invalid rows in C.

   (b) Write a matlab function to implement the flow chart.

4. * Let C denote a 4-by-4 matrix with elements belonging $\{'1', '2', '3', '4'\}$. A column in C is valid if it exactly contains four different elements.

   (a) Draw a flow chart to count the number of invalid columns in C.

   (b) Write a matlab function to implement the flow chart.

5. * Let C denote a 4-by-4 matrix with elements belonging $\{'1','2','3','4'\}$. Let

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix},$$

where $C_1, ..., C_4$ denote four blocks of matrix C. A block of C is valid if its four elements are different.

   (a) Draw a flow chart to count the number of invalid blocks in C.

   (b) Write a matlab function to implement the flow chart.

6. Let f_n denote the Fibonacci series,

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2}, \text{ if } n \geq 2 \\ f_0 &= 0, f_1 = 1. \end{aligned}$$

   (a) Draw a flow chart to calculate f_n for given n.

   (b) Write a matlab function to implement the flow chart.

7. Let A be an n-by-n matrix and $\det(A)$ denote the determinant of matrix A. Let B_{1i} denote a submatrix of A which is obtained by deleting elements in the first row and the ith column of matrix A.

   (a) Write a matlab function to determine B_{1i} for given A and i.

   (b) Express $\det(A)$ in combination of $\det(B_{1i})$ with i running from 1 to n.

   (c) Draw a flow chart to calculate $\det(A)$.

   (d) Write a matlab function to implement the flow chart.

8. Let f be a continuous function. If $f(a)f(b) < 0$ and $a < b$, there exits at least one zero c such that $f(c) = 0$, where $c \in [a, b]$, and c is termed as a zero of f.

   (a) Draw a flow char to seek a zero by binary search for given a, b and f.

   (b) Write a MATLAB function to implement the flow chart.

9. * Let stk be a matlab structure that realizes a stack. Let stk contain two fields. One is an integer array and the other is a non-negative integer. Write a matlab function to implement the following pseudo function for stack allocation.

   (a) Function head: stk=new_stack()

   (b) Function body:

   Set stk.s to an empty vector.

   Set stk.top to zero

10. * Let c denote an integer and stk denote a stack allocated by function new_stack(). Implement the following pseudo function to push c to stack stk.

    (a) Function head: stk=push(c,stk)

    (b) Function body:

    Increase stk.top by one

    n=stk.top

    store c to the nth element of stk.s

11. * Write a matlab function to implement the following pseudo function to check empty of a stack

    (a) Function head: flag=is_empty (stk)

    (b) Function body:

    flag is set to true if stk.top equals zero, and is set to false otherwise

12. * Let stk denote a stack and n equal stk.top. c denotes the nth element of stk.s if n is greater than 0, and is set -1000 otherwise. Implement the following pseudo function to pop c from stack stk.

    (a) Function head: [c,stk]=pop_stack(stk)

    (b) Function body:

    $n = stk.top$

    If n is greater than zero

        Store the nth element of stk.s to c

        Decrease stack.top by one

    Otherwise store -1000 to c

13. * Let o denote a character. In matlab, two characters can be substracted from each other. If o belongs $\{'0', \ldots, '9'\}$, $o -' 0'$ must be within $[0, 9]$. This fact can be applied to check if a given character represents a digit. Write matlab codes to implement the following pseudo function.

    (a) Function head:flag=is_digit(o)

    (b) Function body:

    If o belongs $\{'0', \ldots, '9'\}$

        flag = 1

    Otherwise

        flag = 0

14. * Check whether a given character is an operator.

    (a) Function head: flag=is_operator(o)

    (b) Function body:

    If o is within $\{'+', '-', '*', '/'\}$
        flag = 1
    Otherwise
        flag = 0

15. * Let o be a character belonging $\{'+', '-', '*', '/'\}$ and a and b are two integers. Implement the following pseudo function to apply operator o to a and b. Note that a / b should return floor(a/b).

    (a) Function head: v=evaluate(o, a, b)

    (b) Function body: please refer to lecture slides.

16. * Implement the following pseudo function to evaluate a prefix expression, e.g. *3-5/42=3*(5-4/2)

    (a) Function head: v=prefix(ss)

    (b) Function body

    Allocate stk by function new_stack
    $n = length(ss)$
    for j=n:-1:1
        store the jth element of ss to o
        If o is a digit, translate it to an integer c and push it to stk.
        If o is an operator
            If stk.top is greater than two
                pop an element from stk and store it to integer a
                pop an element from stk and set it to integer b
                apply operator o to a and b and store the result to v
                push v to stack stk
            otherwise display('an invalid prefix expression') and return
        end

17. Describe the Hanoi Tower Problem (HTP). Write a matlab function to solve the HTP problem.

18. Let X be an $N \times 2$ matrix, Y be an $M \times 2$ matrix and D be an $N \times M$ matrix, where $D(i,j) = sqrt((X(i,;)-Y(j,:)).\hat{}2)$ is defined to measure the distance between $X(i,:)$ and $Y(j,:)$. Write a matlab function to determine D for given X and Y by two different approaches.

    (a) Use nested for-loops to determine D for given X and Y.

    (b) Use no for-loops (vector codes) to determine D for given X and Y.

19. Draw a flow chart to illustrate the K-means algorithm for seeking centers of K clusters of given data. Explain the flow chart briefly.

20. * A hyper-plane can be expressed by

$$y = a_1x_1 + a_2x_2 + ... + a_dx_d + b. \text{ eq(1)}$$

Let $a = [a_1, ..., a_d]$. Implement the following pseudo function to get a sample from a hyper-plane defined by a and b.

(a) Head: function [x y] =HPsampling(a,b,n)

(b) Body

d = length(a)

Use rand to generate an $n \times d$ matrix and store it to matrix x.

Rescale elements in x within [-1,1]

Substitute each row of x to eq (1) to attain an element in y.

y = y + rand(n,1)*0.02-0.01.

21. Let x be an $n \times d$ matrix, y be an $n \times 1$ vector and $a = [a_1, ..., a_d]$. The mean square error is defined by

$$E(a, b) = \frac{1}{n}\sum_i (y[i] - (x_i a^T + b))^2,$$

where x_i denote the ith row of matrix x and $y[i]$ denote the ith element of y. Implement the following pseudo function to calculate an mean square error,

(a) Head: function e = mse(x, y, a, b)

(b) Body

n = length(y)

Substitute each row of x to eq (1) to attain an element in vector y_hat

Calculate the mean of the square error between y and y_hat

22. Let $\widehat{a}$ and $\widehat{b}$ minimize $E(a, b)$ of eq (2). Then

$$(\widehat{a}, \widehat{b}) = \arg\min_{a,b} E(a, b).$$

(a) Derive $\widehat{a}$ and $\widehat{b}$ for given x and y.

(b) Write vector codes to determine $\widehat{a}$ and $\widehat{b}$ for given x and y.

23. * Write matlab scripts to accomplish the following tasks.

(a) (10 pts) Let A denote a gray facial image.

1. Rotate image A 90^0 clockwise and display the rotated image.
2. Display the vertical mirror of image A.

(b) (10 pts) Plot the following functions.

1. $cos(x)$ and $sin(x)$ in the same figure within $[-2\pi, 2\pi]$.
2. $cos(x)$ and $sin(x)$ in two subplots within $[-2\pi, 2\pi]$.

(c) (10 pts) Use instruction 'find' to accomplish the following tasks.

1. Generate a uniform sample from [-1 1]x[-1 1]. Plot points of quadrants I and III in red dots and the remains in blue dots.
2. plot the following piecewise linear function

$$\begin{aligned} f(x) &= 2, \quad \text{if } x > 3 \\ &= x - 1, \text{ if } 3 \geq x > 0 \\ &= -x - 1, \quad \text{if } 0 \geq x, \end{aligned}$$

where x is within [-5 5].

24. (10 pts) Solve

$$\begin{aligned} 3x + y - z &= 1 \\ -3x - 2y + 4z &= 0 \\ x + y + z &= 5 \end{aligned}$$

25. (10 pts) Give matlab instructions to generate the following matrices

$$A = \begin{bmatrix} 1 & 4 & 7 & 10 & & 298 \\ 2 & 5 & 8 & 11 & ... & 299 \\ 3 & 6 & 9 & 12 & & 300 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6 & 4 & 5 & 6 & 4 & 5 & 6 \\ 7 & 8 & 9 & 7 & 8 & 9 & 7 & 8 & 9 & 7 & 8 & 9 \end{bmatrix}$$

(a) (10 pts) Let $\mathbf{x} = [x_1, ..., x_N]$, $\mathbf{y} = [y_1, ..., y_N]$ and (x_i, y_i) represent a point in a plane.

1. Form $\mathbf{A}$ and $\mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & N \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} \sum_{i=1}^{N} x_i y_i \\ \sum_{i=1}^{N} y_i \end{bmatrix}.$$

2. Find vector $\mathbf{c}$ such that $\mathbf{Ac} = \mathbf{b}$.

26. * (20 pts) Let $\mathbf{p} = [p_1, p_2, p_3, p_4]$ be a vector whose four elements respectively denote probabilities of generating characters 'A', 'T','C', and 'G'.

(a) 1. Use a for-loop to form $\mathbf{c} = [c_1, c_2, c_3, c_4]$, where

$$c_i = \sum_{j=1}^{i} p_j.$$

2. Use rand to generate a number r within [0,1] and apply the following rule to generate a character,

$$\begin{aligned} R(r) &= \text{'A'}, \ if \qquad r \le c_1 \\ &= \text{'T'}, \ if \ c_1 < r \le c_2 \\ &= \text{'C'}, \ if \ c_2 < r \le c_3 \\ &= \text{'G'}, \ if \ c_3 < r \le c_4 \end{aligned}$$

3. Write a for-loop to generate a string S whose characters are created by rule R.
4. Let $\mathbf{h} = [h_1, h_2, h_3, h_4]$. Its four elements respectively denote occurrences of 'A', 'T','C', and 'G' in a string S. Determine $\mathbf{h}$ for given S.

27. (10 pts) Let A and B denote two N-by-N matrices and $C = AB$. Use nested for-loops to determine C for given A and B.Write matlab functions to accomplish the following tasks.

(a) Calculate the area of an ellipse for given lengths of semi-major axis and semi-minor axis.

(b) Determine a_n for given n, where

$$\begin{aligned} a_n &= 2a_{n-1} + a_{n-2}, \text{ if } n > 1 \\ a_1 &= 1, a_0 = 1 \end{aligned}$$

(c) Let $\mathbf{x} = [x_1, x_2]^T$ and $F(\mathbf{x}) = [f_1(x_1, x_2), f_2(x_1, x_2)]^T$, where

$$\begin{aligned} f_1(x_1, x_2) &= 3x_1^2 + x_2^2 - 16 \\ f_2(x_1, x_2) &= x_1^2 - x_2^2 + 12, \end{aligned}$$

Determine $\mathbf{F}(\mathbf{x})$ for given $\mathbf{x}$.

(d) Calculate $n!$ for given n.

(e) Determine S_N for given N, where

$$S_N = \sum_{n=1}^{N} (\left\lfloor \frac{n^3}{5} \right\rfloor + \left\lceil \frac{2 * n^2}{3} \right\rceil).$$