# Binomial theorem
# Pascal Triangular
# Generating Function
# 2-8

Sequence

# Power Series

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

- A very powerful enumerative tool
- encoding the values of a sequence

$$\{a_n : n \geq 0\}$$

- they can be manipulated just like ordinary functions, i.e., they can be added, subtracted and multiplied, and for our purposes, we generally will not care if the power series con-verges,

# Basic Notation and Terminolog

With a sequence $\sigma = \{a_n : n \geq 0\}$ of real numbers, we associate a "function" $F(x)$ defined by

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

- The word "function" is put in quotes as we do not necessarily care about substituting

- a value of x and obtaining a specific value for F(x). In other words, we consider F(x)

- as a formal power series and frequently ignore issues of convergence

# Example

$$\sigma = \{a_n : n \geq 0\} \text{ with } a_n = 1$$

- F(x)=?

# Constant Sequence

$$F(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$$

Operations:Addition, Subtraction and multiplication

$$\bar{F}(x) = 1 + x + x^2 + x^3 + \cdots$$

$$- \; x\bar{F}(x) = \quad x + x^2 + x^3 + \cdots$$

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$$(1-x)\bar{F}(x) = 1$$

$$\therefore \bar{F}(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

# Maclaurin series

$$F(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

$$F'(x) = 1 + 2x + 3x^2 + \cdots$$

- $$= \sum_{n=1}^{\infty} n x^{n-1}$$

$$= \frac{0 - (-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$F(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$F(-x) = \frac{1}{1+x} = 1 - x + x^2 + \cdots$$

$$\int F(-x)\,dx = \log(1+x)$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots$$

# Product of Two Power Sequences

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n$$

$$C(x) = A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

# Example

- how many ways are there to distribute n apples to one child so that each child receives at least one apple?

- Let $a_n$ denote the answer

$F(x) = x + x^2 + x^3 + \cdots$

denotes a generating function for $\{a_n : n \geq 1\}$

$$F(x) = x(1 + x^2 + x^3 + \cdots) = \frac{x}{1-x}$$

$$(x+x^2+\cdots)(x+x^2+\cdots)(x+x^2+\cdots)(x+x^2+\cdots)(x+x^2+\cdots)$$

$$a_6 x^6 = C\binom{5}{4}$$

$$a_n x^n = a_n x^{k_1+k_2+k_3+k_4+k_5}$$

where $n = k_1 + k_2 + k_3 + k_4 + k_5$

$$k_1, k_2, k_3, k_4, k_5 \geq 1$$

- Returning to the general question here, we're really dealing with distributing n apples to 5 children, and since $k_i>0$ for $i=1,2,...,5$, we also have the guarantee that each child receives at least one apple, so the product of the generating function for one child gives the generating function for five children

$$(x+x^2+\cdots)(x+x^2+\cdots)(x+x^2+\cdots)(x+x^2+\cdots)(x+x^2+\cdots$$

$$= \frac{x^5}{(1-x)^5} = \frac{x^5}{4!}\frac{d^4}{dx^4}\left(\frac{1}{1-x}\right)$$

$$= \frac{x^5}{4!}\frac{d^4}{dx^4}\sum_{n=0}^{\infty} x^n$$

$$= \frac{x^5}{4!}\sum_{n=0}^{\infty} n(n-1)(n-2)(n-3)x^{n-4}$$

$$= \sum_{n=0}^{\infty}\frac{n(n-1)(n-2)(n-3)}{4!}x^{n+1} = \sum_{n=0}^{\infty}\binom{n}{4}x^{n+1}$$

coefficient on $x^n$ in this series $C(n-1,4)$,

# Example

- A grocery store is preparing holiday fruit baskets for sale. Each fruit bas-ket will have 20 pieces of fruit in it, chosen from apples, pears, oranges, and grapefruit. How many different ways can such a basket be prepared if there must be at least one apple in a basket, a basket cannot contain more than three pears, and the number of oranges must be a multiple of four at least one Apple

$$\frac{x}{1-x}$$

three pears at most

$$1+x+x^2+x^3$$

unrestricted grapefruits

$$\frac{1}{1-x}$$

0, 4, 8, 12, ... oranges

$$\frac{1}{1-x^4}$$

$$\frac{x}{1-x}(1+x+x^2+x^3)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x}\right)$$

$$= \frac{x}{1-x}\,\frac{(1-x^4)}{1-x}\,\frac{1}{1-x^4}\,\frac{1}{1-x}$$

$$= x \cdot \frac{1}{(1-x)^3} = \frac{x}{2}\frac{d^2}{dx^2}\frac{1}{1-x}$$

$$\frac{x}{2} \frac{d^2}{dx^2} \sum_{n=0}^{\infty} x^n$$

$$= \frac{x}{2} \sum_{n=0}^{\infty} n(n-1) x^{n-2}$$

$$= \sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^{n-1} = \sum_{n=0}^{\infty} \binom{n}{2} x^{n-1}$$

Answer: C(n+1,2)

# Example

- Find the number of integer solutions to the equation

$$x_1 + x_2 + x_3 = n$$

($n \geq 0$ an integer) with $x_1 \geq 0$ even, $x_2 \geq 0$, and $0 \leq x_3 \leq 2$.

generating functions are

$\frac{1}{1-x^2}$ for $x_1$

$\frac{1}{1-x}$ for $x_2$

$1+x+x^2$ for $x_3$

$$\frac{1+x+x^2}{(1-x)(1-x^2)} = \frac{1+x+x^2}{(1+x)(1-x)^2}.$$

$$\frac{1+x+x^2}{(1+x)(1-x)^2} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

$$1+x+x^2 = A(1-x)^2 + B(1-x^2) + C(1+x).$$

$$1 = A+B+C$$

$$1 = -2A+C$$

$$1 = A-B$$

$A = 1/4$, $B = -3/4$, and $C = 3/2$.

$$\frac{1}{4}\frac{1}{1+x} - \frac{3}{4}\frac{1}{1-x} + \frac{3}{2}\frac{1}{(1-x)^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n - \frac{3}{4}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty} n x^{n-1}.$$

$$\frac{(-1)^n}{4} - \frac{3}{4} + \frac{3(n+1)}{2},$$

# The Binomial Theorem

- The coefficient of $x^{n-k} y^k$ in the expansion
- of $(x+y)^n$ is $\binom{n}{k}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

# Example

x=1, y=1

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

# Pascal's Triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

# Identities in Pascal's Triangle

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

- 

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n}$$

$$= \quad \binom{n-1}{0} - \left[\binom{n-1}{0} + \binom{n-1}{1}\right] + \left[\binom{n-1}{1} + \binom{n-1}{2}\right]$$

$$+ \cdots + (-1)^{n-1} \left[\binom{n-1}{n-2} + \binom{n-1}{n-1}\right] + (-1)^n \binom{n-1}{n-1},$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0,$$

$$
\begin{aligned}
1^2 &= 1,\\
1^2 + 1^2 &= 2,\\
1^2 + 2^2 + 1^2 &= 6,\\
1^2 + 3^2 + 3^2 + 1^2 &= 20,\\
1^2 + 4^2 + 6^2 + 4^2 + 1^2 &= 70.
\end{aligned}
$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}.$$

# Proof

$$S = \{1, 2, 3, \cdots, 2n\}$$

$$= \{1, 2, 3, \ldots, n\} \cup \{n+1, \cdots, 2n\}$$

Selecting n elements from S is considered as selecting k elements from the first half set and n-k elements from the second half set

$$\binom{2n}{n}$$

$$= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} \cdots$$

$$+ \binom{n}{k}\binom{n}{n-k} + \cdots + \binom{n}{n}\binom{n}{0}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k}$$

# Newton's Binomial Theorem

$$(1+x)^p = \sum_{n=0}^{p} \binom{p}{n} x^n.$$

- what happens if we encounter (1+x)^p as a generating function with p not a positive integer

# Recursive Definition

$$P(p,0) = 1$$
$$P(p,k) = pP(p-1,k-1)$$

when $p \geq k > 0$ ($k$ an integer).

$$\binom{p}{k} = \frac{P(p,k)}{k!},$$

# Definition

- Definition For all real numbers p and nonnegative integers k, the number $P(p,k)$ is defined by
- 1. $P(p,0) = 1$ for all real numbers p and
- 2. $P(p,k) = pP(p-1,k-1)$ for all real numbers p and integers $k>0$

# Definition

- Definition. For all real numbers p and nonnegative integers k

- Note that $P(p,k) = C(p,k) = 0$ when p and k are integers with $0 \leq p < k$. On the other hand, we have some interesting new concepts such as $P(-5,4) = (-5)(-6)(-7)(-8)$

$$\binom{-7/2}{5} = \frac{(-7/2)(-9/2)(-11/2)(-13/2)(-15/2)}{5!}.$$

**Theorem 8.9.** *For all real p with $p \neq 0$,*

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n.$$