Binomial theorem Pascal Triangular Generating Function 2-8

Sequence

Power Series

$F(x) = \sum_{n=0}^{\infty} a_n x^n$

- A very powerful enumerative tool
- encoding the values of a sequence

$$\{a_n:n\geq 0\}$$

 they can be manipulated just like ordinary functions, i.e., they can be added, subtracted and multiplied, and for our purposes, we generally will not care if the power series con-verges,

Basic Notation and Terminolog

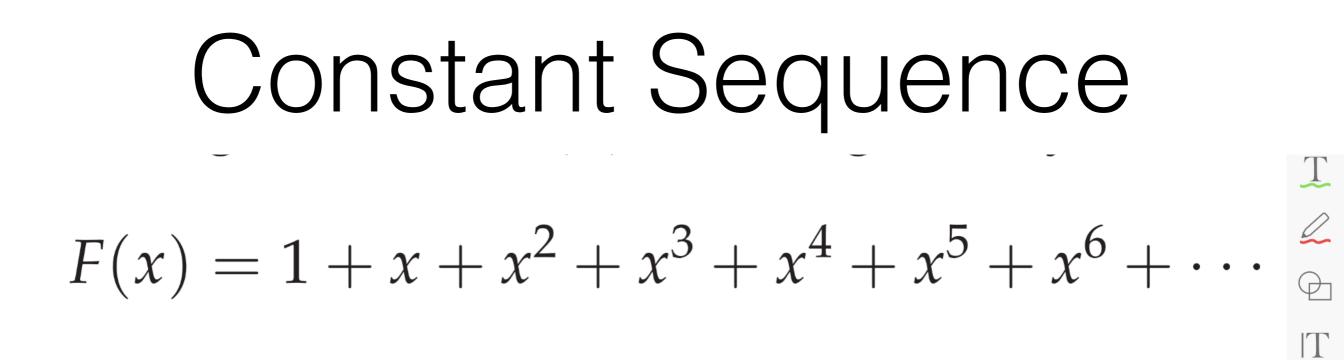
With a sequence $\sigma = \{a_n : n \ge 0\}$ of real numbers, we associate a "function" F(x) defined by

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

- The word "function" is put in quotes as we do not necessarily care about substituting
- a value of x and obtaining a specific value for F(x).
 In other words, we consider F(x)
- as a formal power series and frequently ignore issues of convergence

Example $\sigma = \{a_n : n \ge 0\} \text{ with } a_n = 1 \le 1$

• F(x)=?



Operations: Addition, Subtraction and multiplication

$$\frac{1}{1-(x)} = 1 + \frac{2}{x} + \frac{3}{x} + \frac{3}{x$$

Maclaurin series

 $F(X) = |+X + X + ... = \frac{1}{1-X}$ $\overline{\int} (X) = 1 + 2X + 3X + \cdots$ $=\sum_{n \neq n}^{\infty} n x^{n-1}$ $\Lambda \equiv 1$ $\frac{D}{2} - (-1) = \frac{1}{(1-\chi)^2}$

$$F(x) = \frac{1}{1-x} = 1 + x + x + \cdots$$

$$F(-x) = \frac{1}{1+x} = 1 - x + x + \cdots$$

$$\int F(-x) dx = \log(1+x)$$

= $x - \frac{1}{2}x^{2} + \frac{1}{3}x^{3} - \frac{1}{4}x^{4}$

Product of Two Power Sequences

Example

 how many ways are there to distribute n apples to one child so that each child receives at least one apple?

• Let a._n denote the answer

$$F(x) = X + X^{2} + X^{3} + \cdots$$

denotes a generating
function for $\{a_{n} : n \ge 1\}$
$$F(x) = X(1 + X + X^{3} + \cdots) = \frac{X}{1 - X}$$

(X + X + ...)(X + ...)(X + X + ...)(X $\mathcal{A}_{\mathcal{A}}^{\mathsf{F}} = C\left(\begin{array}{c} 5\\ 4\end{array}\right)$ $Q_{\eta}X^{n} = Q_{\eta}X^{k_{1}+k_{2}+k_{3}+k_{4}+k_{5}}$ where $N = K_1 + K_2 + K_3 + K_4 + K_5$ $|k_1, k_2, k_3, k_4, k_5 > |$

 Returning to the general question here, we're really dealing with distributing n apples to 5 children, and since ki>0 for i=1,2,...,5, we also have the guarantee that each child receives at least one apple, so the product of the generating function for one child gives the generating function for five children

(x + x + y)(x + x + $= \frac{x^{5}}{(1-X)^{5}} = \frac{x^{5}}{41} \frac{d^{4}}{dx^{4}} \left(\frac{1}{1-x}\right)$ $= \frac{\chi^5}{4!} \frac{d^4}{d\chi^4} \sum_{n=0}^{\infty} \chi^n$ $= \frac{X}{4!} \sum_{n=0}^{\infty} n(n-1)(n-2)(n-3)X$ $= \sum_{n=0}^{\infty} \frac{n(n-1)(n-2)(n-3)}{4!} X^{n+1} = \sum_{n=0}^{\infty} {n \choose 4} x^{n+1}$

coefficient on x^n in this series C(n - 1, 4),

Example

 A grocery store is preparing holiday fruit baskets for sale. Each fruit bas-ket will have 20 pieces of fruit in it, chosen from apples, pears, oranges, and grapefruit. How many different ways can such a basket be prepared if there must be at least one apple in a basket, a basket cannot contain more than three pears, and the number of oranges must be a multiple of four

at least one Apple $\frac{X}{I-X}$ three pears at most $|+X+X+X^{2}$ unrestricted grapefoaits $\frac{1}{1 - X}$ 0.4,8,12,... Dranges | | -×4

 $\frac{\chi}{1-\chi}\left(1+\chi+\chi+\chi^{2}+\chi^{3}\right)\left(\frac{1}{1-\chi^{4}}\right)\left(\frac{1}{1-\chi}\right)$ $= \frac{x}{1-x} \frac{(1-x^{4})}{1-x} \frac{1}{1-x^{4}} \frac{1}{1-x}$ $= X \cdot \frac{1}{(1-x)^{3}} = \frac{x}{z} \frac{d^{2}}{dx^{2}} \frac{1}{1-x}$

 $\frac{\chi}{z} = \frac{d}{d\chi^2} \sum_{n=0}^{\infty} \chi^n$ $=\frac{1}{2}\sum_{n=1}^{\infty}n(n-1)X^{n-2}$ $= \sum_{n=0}^{n=0} \frac{n(n-1)}{2} \chi^{n-1} = \sum_{n=-1}^{\infty} {\binom{n}{2} \chi^{n-1}} = \sum_{n=-1}^{\infty} {\binom{n}{2} \chi^{$

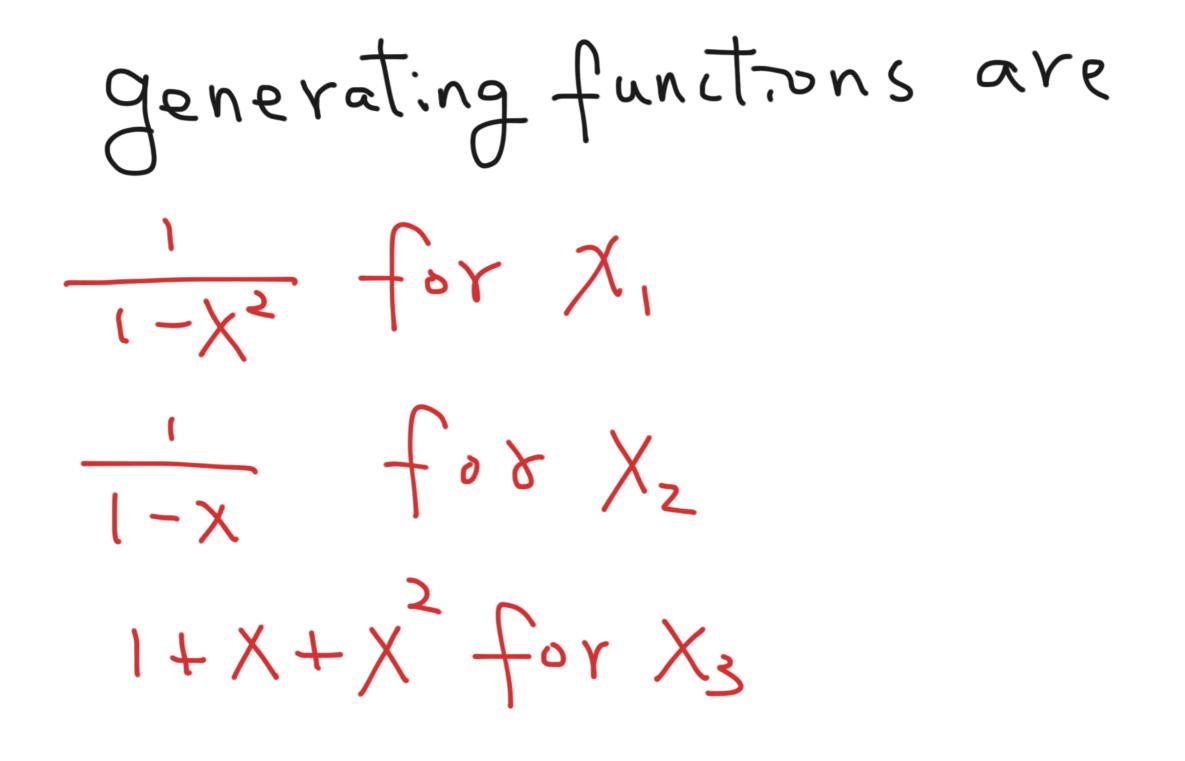
Answer: C(n+1,2)

Example

• Find the number of integer solutions to the equation

 $x_1 + x_2 + x_3 = n$

($n \ge 0$ an integer) with $x_1 \ge 0$ even, $x_2 \ge 0$, and $0 \le x_3 \le 2$.



$$\frac{1+x+x^2}{(1-x)(1-x^2)} = \frac{1+x+x^2}{(1+x)(1-x)^2}.$$
$$\frac{1+x+x^2}{(1+x)(1-x)^2} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

$$1 + x + x^{2} = A(1 - x)^{2} + B(1 - x^{2}) + C(1 + x).$$

$$1 = A + B + C$$

$$1 = -2A + C$$

$$1 = A - B$$

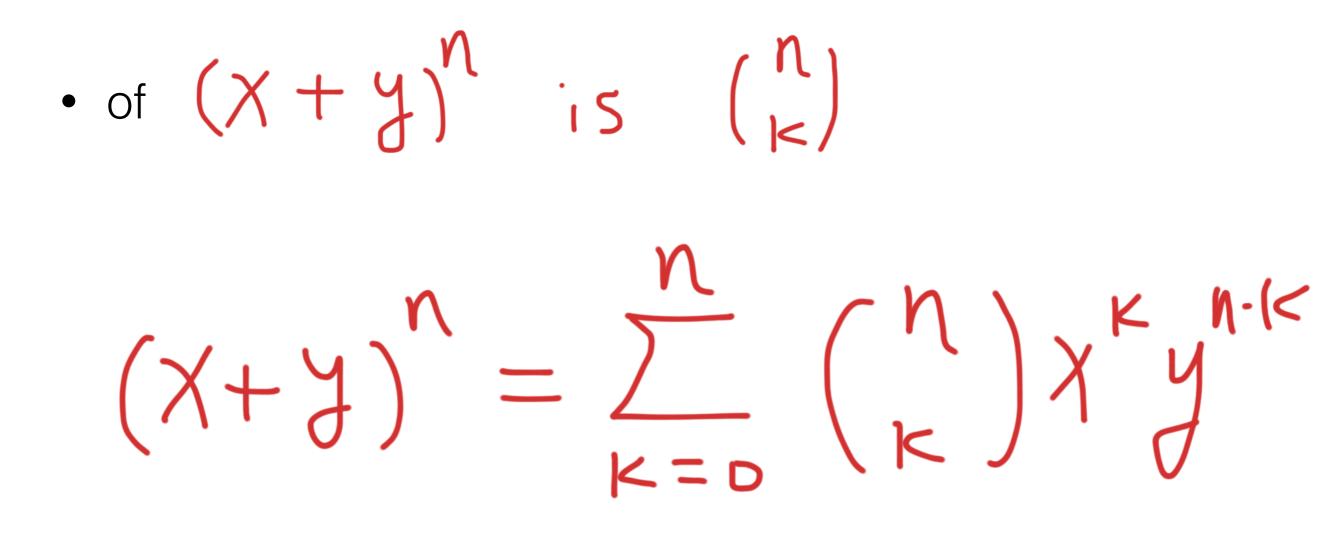
A = 1/4, B = -3/4, and C = 3/2.

$$\frac{1}{4}\frac{1}{1+x} - \frac{3}{4}\frac{1}{1-x} + \frac{3}{2}\frac{1}{(1-x)^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n - \frac{3}{4}\sum_{n=0}^{\infty}x^n + \frac{3}{2}\sum_{n=0}^{\infty}nx^{n-1}.$$

$$\frac{(-1)^n}{4} - \frac{3}{4} + \frac{3(n+1)}{2},$$

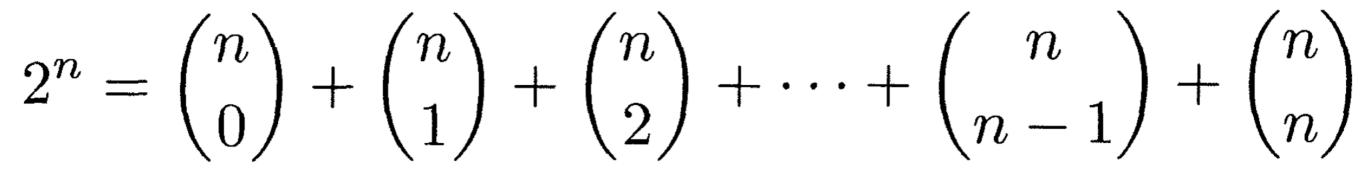
The Binomial Theorem

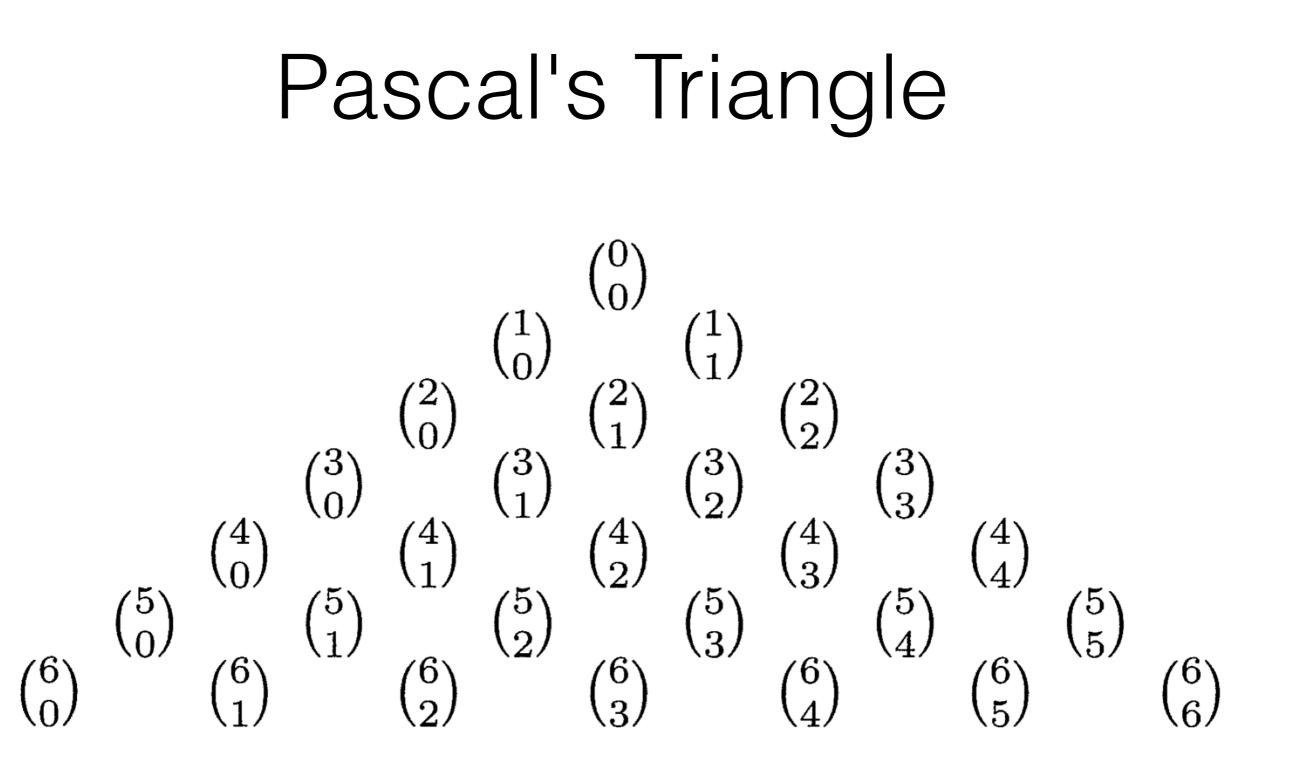
• The coefficient of $\chi^{n-k} \Upsilon^{k}$ in the expansion



Example

x = 1, y = 1





 $\mathbf{2}$ $15 \quad 20 \quad 15$ $7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7$ 56 70 56

Identities in Pascal's Triangle

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n \binom{n}{n}$$

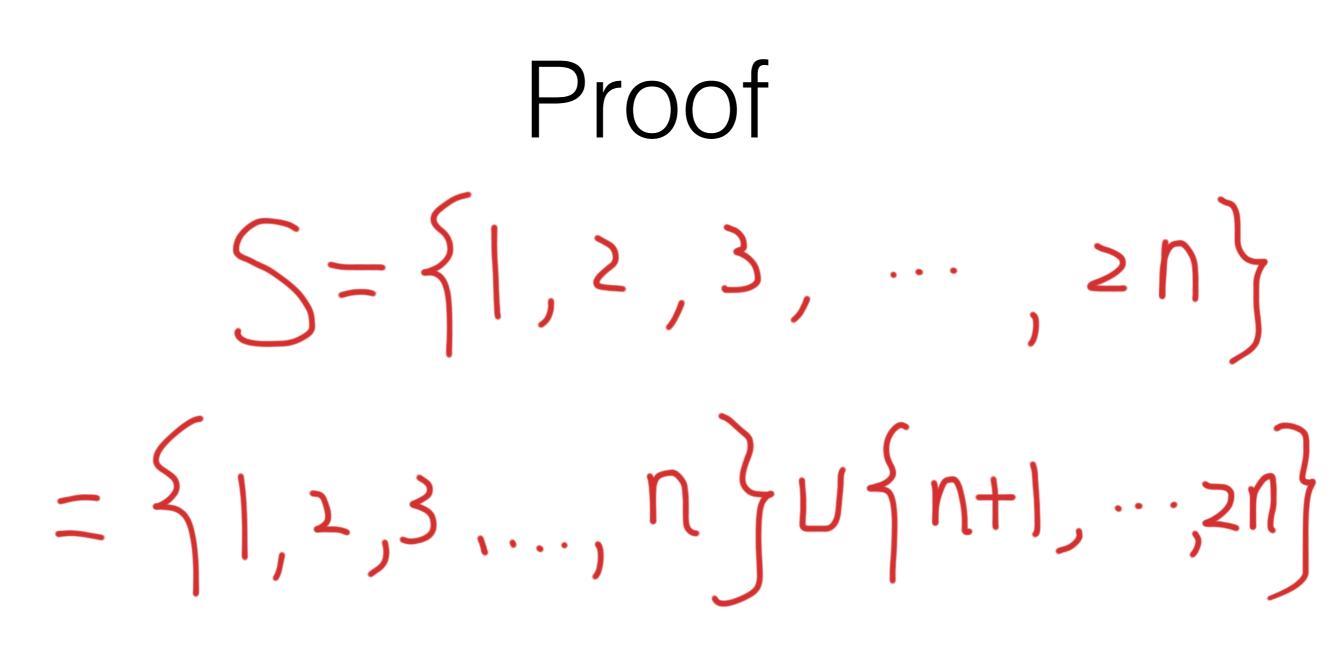
$$= \binom{n-1}{0} - \left[\binom{n-1}{0} + \binom{n-1}{1} \right] + \left[\binom{n-1}{1} + \binom{n-1}{2} \right]$$

$$+\cdots + (-1)^{n-1} \left[\binom{n-1}{n-2} + \binom{n-1}{n-1} \right] + (-1)^n \binom{n-1}{n-1},$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n \binom{n}{n} = 0,$$

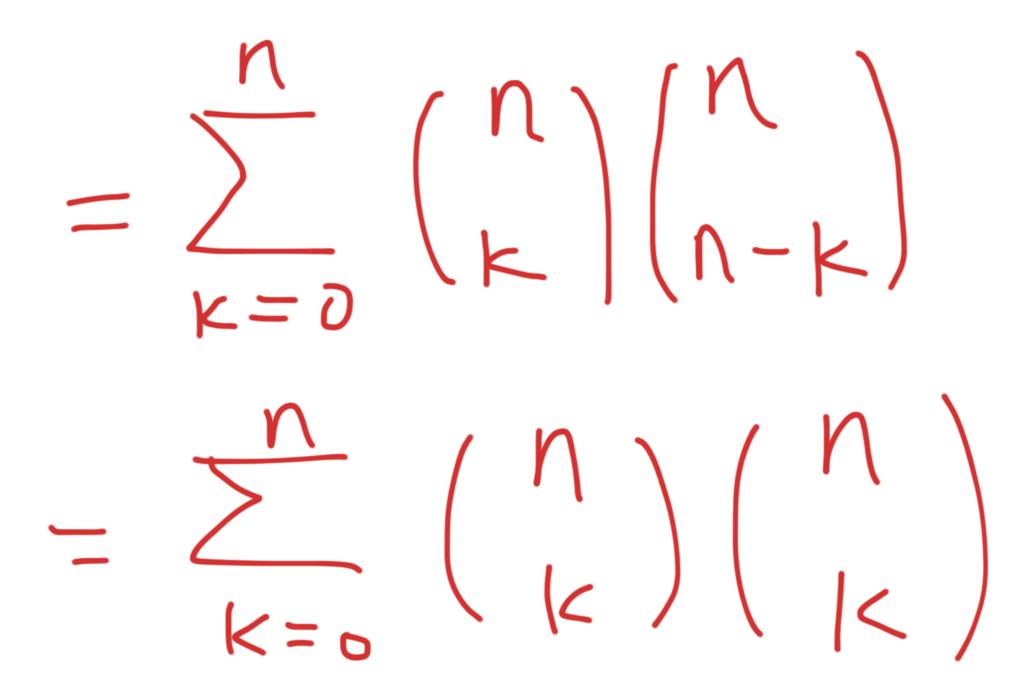
$1^2 = 1,$ $1^2 + 1^2 = 2$. $1^2 + 2^2 + 1^2 = 6.$ $1^2 + 3^2 + 3^2 + 1^2 = 20$, $1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70.$

$$\binom{n}{0}^{2} + \binom{n}{1}^{2} + \binom{n}{2}^{2} + \dots + \binom{n}{n-1}^{2} + \binom{n}{n}^{2} = \binom{2n}{n}$$



Selecting n elements from S is considered as selecting k elements from the first half set and n-k elements from the second half set

 $\left(2n \right)$ $= \binom{n}{n}\binom{n}{n} + \binom{n}{n}\binom{n}{n-1} \cdots$ $+ \binom{n}{k}\binom{n-k}{k-k} + \cdots + \binom{n}{k}\binom{n}{k}$



Newton's Binomial Theorem

 $(1+x)^p = \sum_{n=0}^p \binom{p}{n} x^n.$

 what happens if we encounter (1+x)^p as a generating function with p not a positive integer

Recursive Definition

P(p,0) = 1P(p,k) = pP(p-1,k-1)when $p \ge k > 0$ (*k* an integer).

 $\binom{p}{k} = \frac{P(p,k)}{k!},$

Definition

- Definition For all real numbers p and nonnegative integers k, the number P(p,k) is defined by
- 1. P(p,0) = 1 for all real numbers p and
- 2. P(p,k) =pP(p-1,k-1)for all real numbers p and integers k>0

Definition

- Definition. For all real numbers p and nonnegative integers k
- Note that P(p,k) =C(p,k) =0 when p and k are integers with 0≤p<k. On the other hand, we have some interesting new concepts such as P(-5,4) = (-5)(-6)(-7)(-8)

$$\binom{-7/2}{5} = \frac{(-7/2)(-9/2)(-11/2)(-13/2)(-15/2)}{5!}.$$

Theorem 8.9. For all real p with $p \neq 0$,

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n.$$