Combinatorial Tools

- Induction
- Comparing and Estimating Numbers
- Inclusion-Exclusion
- Pigeonholes
- The Twin Paradox and the Good Old Logarithm

$$
\begin{aligned}
1 &= 1 \\
1 + 3 &= 4 \\
1 + 3 + 5 &= 9 \\
1 + 3 + 5 + 7 &= 16 \\
1 + 3 + 5 + 7 + 9 &= 25 \\
1 + 3 + 5 + 7 + 9 + 11 &= 36 \\
1 + 3 + 5 + 7 + 9 + 11 + 13 &= 49 \\
1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 &= 64 \\
1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 &= 81 \\
1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 &= 100
\end{aligned}
$$

$$1 + 3 + \cdots + (2n - 3) + (2n - 1) = n^2.$$

$$1 + 3 + \cdots + (2n - 3) + (2n - 1) = \Big(1 + 3 + \cdots + (2n - 3)\Big) + (2n - 1).$$

$$(n - 1)^2 + (2n - 1) = (n^2 - 2n + 1) + (2n - 1) = n^2, \tag{2.2}$$

- suppose that we can prove two facts:
- (a) 1 has the property, and
- (b) whenever n -1 has the property, then n also has the property (n > 1).

- The Principle of Induction says that if (a) and (b) are true, then every

- natural number has the property.

- This is precisely what we did above. We showed that the "sum" of the first 1 odd numbers is 12, and then we showed that if the sum of the first n -1 odd numbers is (n -1) 2, then the sum of the first n odd numbers is n2, for whichever integer n > 1 we consider. Therefore, by the Principle of Induction we can conclude that for every positive integer n, the sum of the first n odd numbers is n2.

Comparing and Estimating Numbers

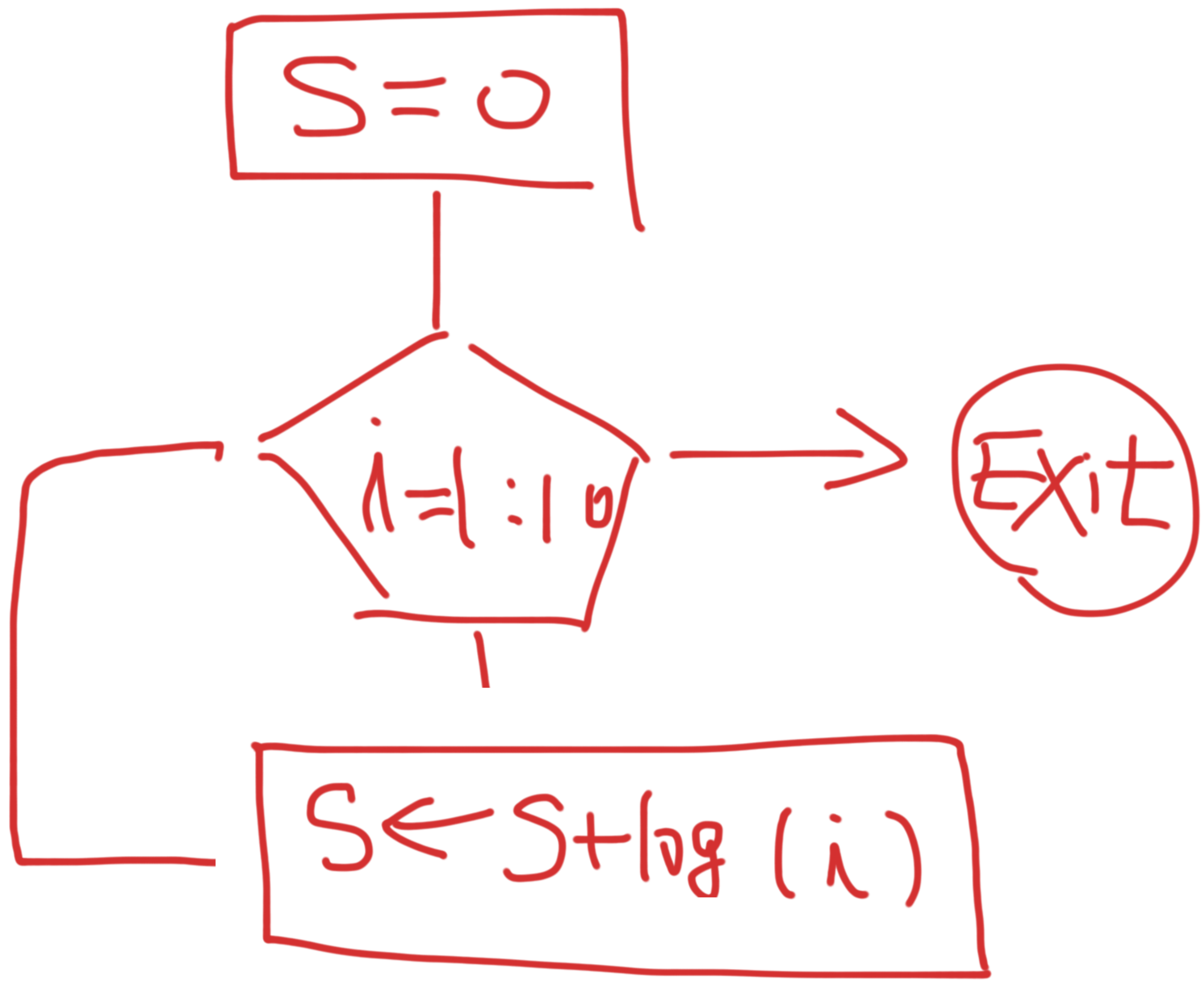
Theorem 1.1. *As* $n \to \infty$, $n! \sim \frac{n^n}{e^n}\sqrt{2\pi n}$. *That is,* $\lim_{n\to\infty} \frac{n!}{(n^n/e^n)\sqrt{2\pi n}} = 1$.

Stirling's formula can also be expressed as an estimate for $\log(n!)$:

$$\log(n!) = n\log n - n + \frac{1}{2}\log n + \frac{1}{2}\log(2\pi) + \varepsilon_n, \tag{1.1}$$

where $\varepsilon_n \to 0$ as $n \to \infty$.

```
>> n=10;a1=n*log(n)-n+log(n)/2+log(2*pi)/2

a1 =

   15.0961

>> s=0;for i=1:10 s=s+log(i); end
>> s

s =

   15.1044
```


S=0
i=1:10
Exit
S←S+log(i)

```
>> n=20;a1=n*log(n)-n+log(n)/2+log(2*pi)/2

a1 =

   42.3315

>> s=0;for i=1:20 s=s+log(i); end
>> s

s =

   42.3356
```

- Taking n = 10, log(10!) ≈ 15.104 and the logarithm of Stirling's approxi- mation to 10! is approximately 15.096, so log(10!) and its Stirling approximation differ by roughly .008.

Inclusion and exclusion

In a class of 40, many students are collecting the pictures of their favorite rock stars. Eighteen students have a picture of the Beatles, 16 students have a picture of the Rolling Stones and 12 students have a picture of Elvis Presley (this happened a long time ago, when we were young). There are 7 students who have pictures of both the Beatles and the Rolling Stones, 5 students who have pictures of both the Beatles and Elvis Presley, and 3 students who have pictures of both the Rolling Stones and Elvis Presley. Finally, there are 2 students who possess pictures of all three groups. Ques-tion: How many students in the class have no picture of any of the rock groups?

Name	Bonus	Beatles	Stones	Elvis	BS	BE	SE	BSE
Al	1	0	0	0	0	0	0	0
Bel	1	-1	0	0	0	0	0	0
Cy	1	-1	-1	0	1	0	0	0
Di	1	-1	0	-1	0	1	0	0
Ed	1	-1	-1	-1	1	1	1	-1
⋮								

TABLE 2.1. Strange record of who's collecting whose pictures.

- First, we give a bonus of 1 to every student.
- Second, we record in a separate column whether the student is collecting (say) both the Beatles and Elvis Presley (the column labeled BE), even though this could be read off from the previous columns.
- Third, we put a -1 in columns recording the collecting of an odd number of pictures, and a 1 in columns recording the collecting of an even number of pictures.

what are the row sums?

We get 1 for Al and 0 for everybody else.

One picture of B, S or E

B	S	E
-1	0	0
-1	-1	0
-1	-1	-1

No picture

1 0 0 0

One picture of B, S or E

Bonus	B	S	E	BS	BE	SE	BSE
1	-1	0	0				
1	-1	-1	0	1			
1	-1	-1	-1	1	1	1	-1

For all three cases, one (-1), two (-1). And three (-1), row sum equals zero

- if a row records at least one -1, it's row sum is zero
- If a row records no picture, It's row sum is one

$$\sum_{i} (row\_sum\_i) = \sum_{j} (column\_sum\_j)$$

A quantity represents the number of students who collect no pictures

Column sums

Name	Bonus	Beatles	Stones	Elvis	BS	BE	SE	BSE
Al	1	0	0	0	0	0	0	0
Bel	1	−1	0	0	0	0	0	0
Cy	1	−1	−1	0	1	0	0	0
Di	1	−1	0	−1	0	1	0	0
Ed	1	−1	−1	−1	1	1	1	−1
⋮								
	40	18	16	12	7	5	3	2

$$\sum_{j} (column\_sum\_j)$$

$$= 40 - 18 - 16 - 12 + 7 + 5 + 3 - 2$$

$$= \sum_{i} (row\_sum\_i)$$

This formula is called the Inclusion-Exclusion Formula or Sieve Formula. The origin of the first name is obvious; the second refers to the image that we start with a large set of objects and then "sieve out" those objects we don't want to count.

Pigeonhole

- If we have n boxes and we place more than n objects into them, then there will be at least one box that contains more than one object.

- Hand-shaking
- If there are n people who can shake hands with one another (where $n > 1$), the pigeonhole principle shows that there is always a pair of people who will shake hands with the same number of people.

- Hand-shaking
- As the 'holes', or m, correspond to number of hands shaken, and each person can shake hands with anybody from 0 to n – 1 other people, this creates n – 1 possible holes. This is because either the '0' or the 'n – 1' hole must be empty (if one person shakes hands with everybody, it's not possible to have another person who shakes hands with nobody; likewise, if one person shakes hands with no one there cannot be a person who shakes hands with everybody). This leaves n people to be placed in at most n – 1 non-empty holes, guaranteeing duplication.

- A probabilistic generalization of the pigeonhole principle states that if n pigeons are randomly put into m pigeonholes with uniform probability 1/m, then at least one pigeonhole will hold more than one pigeon with probability

All possibilities

n objects $\leq m$

m^n

n objects are assigned to distinct holes

$$C_n^m \cdot n!$$

$$= \frac{m \times (m-1) \cdots (m-n+1)}{n!} n!$$

$$= (m)_n$$

$$1 - \frac{(m)_n}{m^n}, \quad n \le m$$

$$(m)_n = m(m-1)\cdots(m-n+1)$$

- Softball team

- Imagine seven people who want to play softball (n = 7 items), with a limitation of only four softball teams (m = 4 holes) to choose from. The pigeonhole principle tells us that they cannot all play for different teams. At least 2 must play on the same team

- or example, if 2 pigeons are randomly assigned to 4 pigeonholes, there is a 25% chance that at least one pigeonhole will hold more than one pigeon; for 5 pigeons and 10 holes, that probability is 69.76%; and for 10 pigeons and 20 holes it is about 93.45%. If the number of holes stays fixed, there is always a greater probability of a pair when you add more pigeons

$m = 4$

$\square \quad \square \quad \cdots \quad \square$

$n = 2$

$$1 - \frac{(m)_n}{m^n} = 1 - \frac{4 \times 3}{4 \times 4}$$

$$= 1 - \frac{3}{4} = 25\%$$

```
>> s='1-factorial(m)/factorial(n)/m^n'

s =

1-factorial(m)/factorial(n)/m^n

>> f=inline(s)

f =

     Inline function:
     f(m,n) = 1-factorial(m)/factorial(n)/m^n

>> f(4,2)

ans =

    0.2500
```

$m = 10$

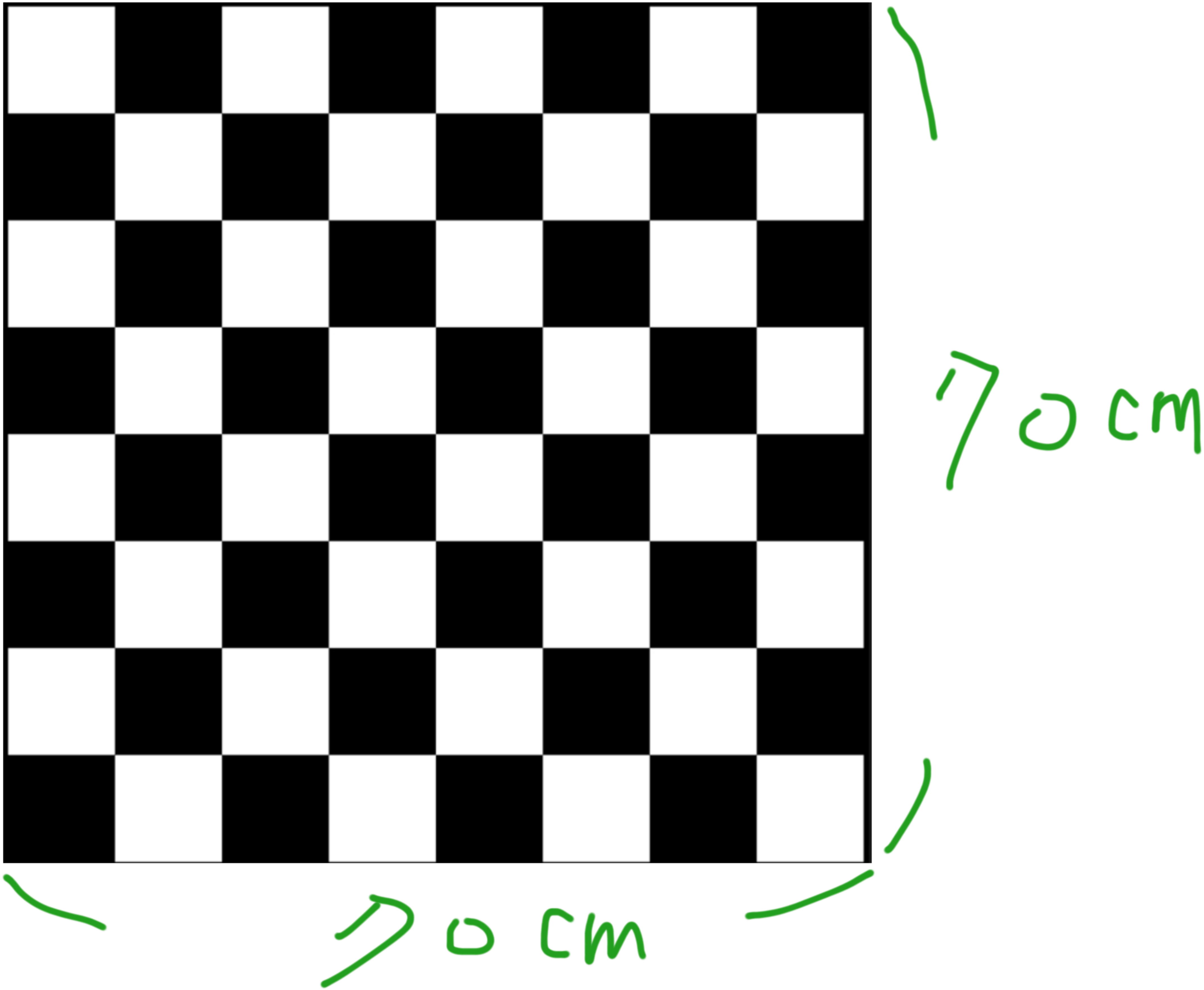
$n = 5$

$$1 - \frac{(m)_n}{m^n} = 1 - \frac{10 \times 9 \times 8 \cdots 6}{10^5} = 69.76\%$$

```
>> f(10,5)

ans =

    0.6976
```

```
>> f(20,10)

ans =

    0.9345
```

- We shoot 50 shots at a square target, the side of which is 70 cm long. We are quite a good shot, because all of our shots hit the target. Prove that there are two bulletholes that are closer than 15 cm.


70 cm
70 cm

There are 50 pigeons and 49 pigeonholes.
There are two pigeons in the same hole

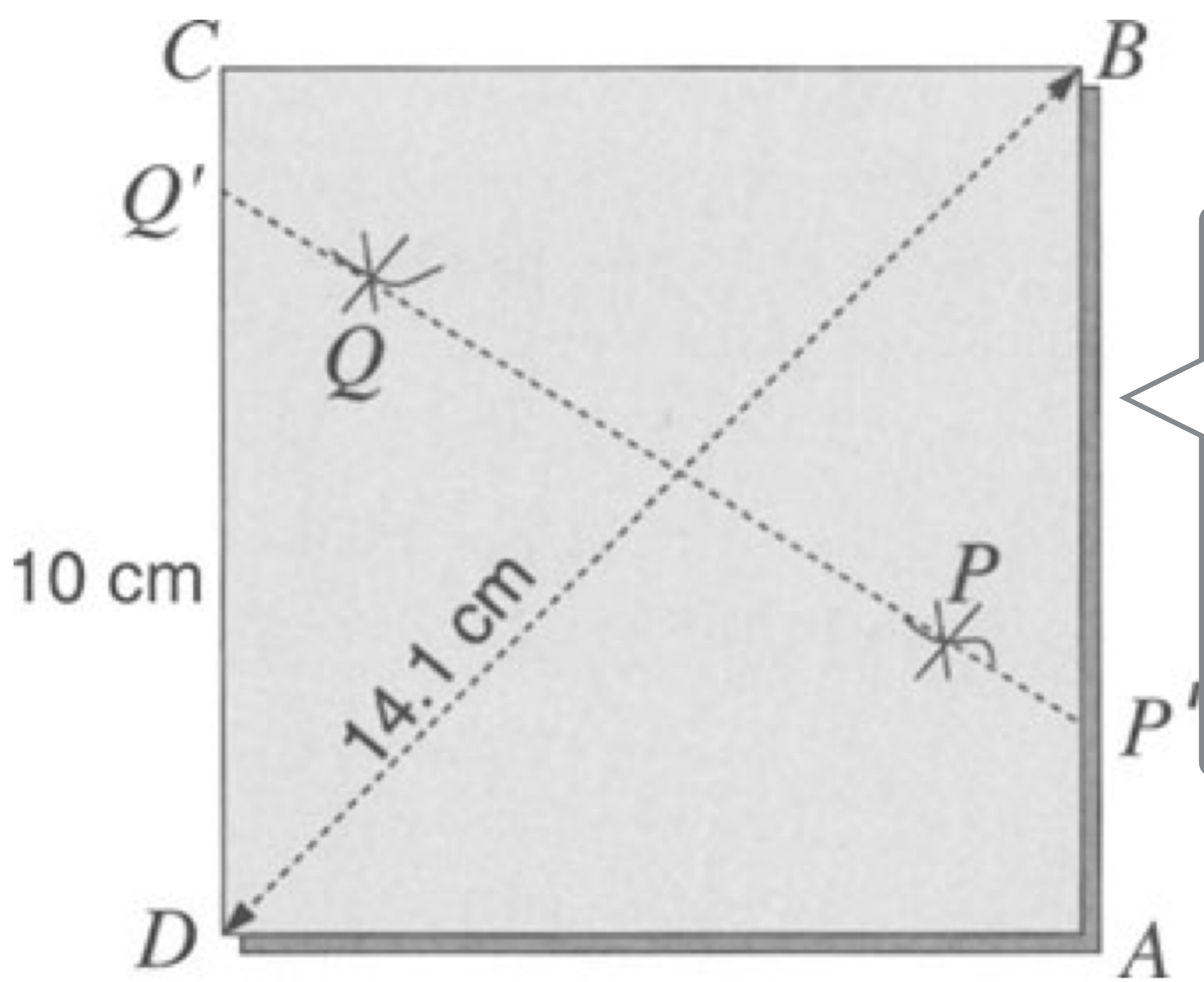


FIGURE 2.2. Two shots in the same square

- two bulletholes in a square must be closer than 15 cm.

The Twin Paradox and the Good Old Logarithm

- I bet that there are two of you who have the same birthday! What do you think?

- "There are 366 possible birthdays and fifty students.
- m=366 pigeonholes and n=50 pigeons. Find the probability of 50 pigeons at distinct pigeonholes.

$$\frac{366 \cdot 365 \cdots 317}{366^{50}}.$$

$$\frac{n^k}{n(n-1)\cdots(n-k+1)} = \frac{n}{n-1} \cdots \frac{n}{n-k+1}.$$

$$\log\left(\frac{n^k}{n(n-1)\cdots(n-k+1)}\right) = \sum_{i=1}^{k-1} \log\frac{n}{n-i}$$

```
>> a=0;for i=1:49 a=a+log(366/(366-i)); end
>> a

a =

    3.5090
```

```
>> 1/exp(a)

ans =

 0.0299
```

$$\frac{366 \cdot 365 \cdots 317}{366^{50}}.$$

```
>> exp(a)

ans =

  33.4147
```

$$28.4 \leq \frac{366^{50}}{366 \cdot 364 \cdots 317} \leq 47.7.$$