Gentzen's deduction system

We use variables for the labels, and a packet labeled with x consisting of occurrences of the proposition P is written as x: P.

Thus, in a sequent $\Gamma \rightarrow P$, the expression Γ is any finite set of the for m x1: P1,...,xm: Pm,where the xi are pairwise distinct (but the Pi need not be dis-tinct). Given Γ = x1: P1,...,xm: Pm, the notation Γ,x: P is only well defined when x $\neq$ xi for all i, $1 \leq i \leq m$, in which case it denotes the set x1: P1,...,xm: Pm,x: P.

A sequent

$$\Gamma \rightarrow P$$

$$\Gamma = x_1 : P_1, x_2 : P_2, \ldots, x_m : P_m$$

Definition 1.2. The axioms and inference rules of the system $\mathcal{NG}_m^{\Rightarrow}$

(implicational
logic, Gentzen-sequent style (the G in N G stands for Gentzen)) are listed below:

RULE I: $\Gamma, x\colon P \to P$ (Axioms)

RULE II:

$$\frac{\Gamma, x\colon P \to Q}{\Gamma \to P \Rightarrow Q} \quad (\Rightarrow\text{-}intro)$$

RULE III:

$$\frac{\Gamma \to P \Rightarrow Q \qquad \Gamma \to P}{\Gamma \to Q} \quad (\Rightarrow\text{-}elim)$$

$$\cfrac{\cfrac{\cfrac{P^x, Q^y}{P}}{Q \Rightarrow P}\; y}{P \Rightarrow (Q \Rightarrow P)}\; x$$

ORIGINAL SYSTEM

$$\cfrac{\cfrac{x\colon P, y\colon Q \to P}{x\colon P \to Q \Rightarrow P}}{\to P \Rightarrow (Q \Rightarrow P)}$$

NEW SYSTEM

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(A \Rightarrow (B \Rightarrow C))^{z} \qquad A^{x}}{B \Rightarrow C} \qquad \dfrac{(A \Rightarrow B)^{y} \qquad A^{x}}{B}}{C}\; x}{A \Rightarrow C}\; y}{(A \Rightarrow B) \Rightarrow (A \Rightarrow C)}\; z}{\big(A \Rightarrow (B \Rightarrow C)\big) \Rightarrow \big((A \Rightarrow B) \Rightarrow (A \Rightarrow C)\big)}$$

$$\Gamma = x : A \Rightarrow (B \Rightarrow C), y : A \Rightarrow B, z : A.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \to A \Rightarrow (B \Rightarrow C) \qquad \Gamma \to A}{\Gamma \to B \Rightarrow C} \qquad \dfrac{\Gamma \to A \Rightarrow B \qquad \Gamma \to A}{\Gamma \to B}}{x : A \Rightarrow (B \Rightarrow C), y : A \Rightarrow B, z : A \to C}}{x : A \Rightarrow (B \Rightarrow C), y : A \Rightarrow B \to A \Rightarrow C}}{x : A \Rightarrow (B \Rightarrow C) \to (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}}{\to \big(A \Rightarrow (B \Rightarrow C)\big) \Rightarrow \big((A \Rightarrow B) \Rightarrow (A \Rightarrow C)\big)}$$

$$\Gamma = x\colon A \Rightarrow (B \Rightarrow C), y\colon A \Rightarrow B, z\colon A.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \to A \Rightarrow (B \Rightarrow C) \qquad \Gamma \to A}{\Gamma \to B \Rightarrow C} \qquad \dfrac{\Gamma \to A \Rightarrow B \qquad \Gamma \to A}{\Gamma \to B}}{x\colon A \Rightarrow (B \Rightarrow C), y\colon A \Rightarrow B, z\colon A \to C}}{x\colon A \Rightarrow (B \Rightarrow C), y\colon A \Rightarrow B \to A \Rightarrow C}}{x\colon A \Rightarrow (B \Rightarrow C) \to (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}}{\to \big(A \Rightarrow (B \Rightarrow C)\big) \Rightarrow \big((A \Rightarrow B) \Rightarrow (A \Rightarrow C)\big)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(A \Rightarrow (B \Rightarrow C))^z \qquad A^x}{B \Rightarrow C} \qquad \dfrac{(A \Rightarrow B)^y \qquad A^x}{B}}{C}}{A \Rightarrow C}\,x}{(A \Rightarrow B) \Rightarrow (A \Rightarrow C)}\,y}{\big(A \Rightarrow (B \Rightarrow C)\big) \Rightarrow \big((A \Rightarrow B) \Rightarrow (A \Rightarrow C)\big)}\,z$$

Elimination rule

The first version

$$\dfrac{\begin{matrix}\Gamma & \Gamma \\ \mathscr{D}_1 & \mathscr{D}_2 \\ P \Rightarrow Q & P\end{matrix}}{Q}$$

The second Version

$$\frac{\Gamma \rightarrow P \Rightarrow Q \quad \Gamma \rightarrow P}{\Gamma \rightarrow Q}$$

A version corresponds definition 1.1

$$\frac{\Gamma \rightarrow P \Rightarrow Q \quad \Delta \rightarrow P}{\Gamma, \Delta \rightarrow Q} \quad (\Rightarrow\text{-}elim'),$$

- Definition 1.3. The axioms, inference rules, and deduction trees for (propositional) classical logic are defined as follows.

- Axioms:

- (i) Every one-node tree labeled with a single proposition P is a deduction tree for P with set of premises {P}

(ii) The tree

$$\frac{\Gamma, P}{P}$$

is a deduction tree for P with multiset of premises $\Gamma \cup \{P\}$.

The ⇒-introduction rule:

If D is a deduction of Q from the premises in Γ ∪ {P}, the is a deduction tree for P ⇒ Q from Γ. All premises P labeled x are discharged

$$\dfrac{\begin{array}{c}\Gamma, P^x \\ \mathscr{D} \\ Q\end{array}}{P \Rightarrow Q}\; x$$

The ⇒-elimination rule (or modus ponens):

If D1 is a deduction tree for P ⇒ Q from the premises Γ, and D2. is a deduction for P from the premises Δ, the

$$\dfrac{\begin{matrix}\Gamma \\ \mathscr{D}_1 \\ P \Rightarrow Q\end{matrix} \qquad \begin{matrix}\Delta \\ \mathscr{D}_2 \\ P\end{matrix}}{Q}$$

is a deduction tree for Q from the premises in Γ ∪ Δ

The $\wedge$-introduction rule:
If D1 is a deduction tree for P from the premises Γ, and D2is deduction for Q
from the premises Δ, then

$$\dfrac{\begin{matrix} \Gamma \\ \mathscr{D}_1 \\ P \end{matrix} \qquad \begin{matrix} \Delta \\ \mathscr{D}_2 \\ Q \end{matrix}}{P \wedge Q}$$

is a deduction tree for $P \wedge Q$ from the premises in $\Gamma \cup \Delta$

The $\wedge$-elimination rule:
If D is a deduction tree for $P \wedge Q$ from the premises Γ, then

$$\dfrac{\begin{matrix}\Gamma \\ \mathscr{D} \\ P \wedge Q\end{matrix}}{P} \qquad\qquad \dfrac{\begin{matrix}\Gamma \\ \mathscr{D} \\ P \wedge Q\end{matrix}}{Q}$$

are deduction trees for P and Q from the premises Γ

The $\vee$-introduction rule:

If D is a deduction tree for P or for Q from the premises Γ, then

$$\frac{\begin{array}{c}\Gamma \\ \mathscr{D} \\ P\end{array}}{P \vee Q} \qquad\qquad \frac{\begin{array}{c}\Gamma \\ \mathscr{D} \\ Q\end{array}}{P \vee Q}$$

are deduction trees for $P \vee Q$ from the premises in Γ

The ⇒-elimination rule (or modus ponens):

If D1 is a deduction tree for P ⇒ Q from the premises Γ,

and D2 is a deduction
for P from the premises Δ, then

$$\dfrac{\begin{matrix}\Gamma & \Delta, P^x & \Lambda, Q^y \\ \mathscr{D}_1 & \mathscr{D}_2 & \mathscr{D}_3 \\ P \vee Q & R & R\end{matrix}}{R}\quad x,y$$

is a deduction tree for Q from the premises in Γ ∪ Δ ∪ Λ.
All premises P labeled x
and all premises Q labeled y are discharged

The $\perp$-elimination rule:
If D is a deduction tree for $\perp$ from the premises Γ, then

$$\frac{\begin{array}{c}\Gamma \\ \mathscr{D} \\ \perp\end{array}}{P}$$

is a deduction tree for P from the premises Γ, for any proposition P

The proof-by-contradiction rule (also known as reductio ad absurdum rule, for
short RAA):
If D is a deduction tree for ⊥ from the premises in Γ ∪{¬P}, then

$$\dfrac{\begin{matrix} \Gamma, \neg P^{x} \\ \mathscr{D} \\ \perp \end{matrix}}{P}\; x$$

is a deduction tree for P from the premises Γ. All premises ¬P labeled x, are dis-
charged.

Because $\neg P$ is an abbreviation for $P \Rightarrow \bot$,

the $\neg$-introduction rule is a special case of the $\Rightarrow$-introduction rule (with $Q = \bot$).

The ¬-introduction rule:
If D is a deduction tree for $\perp$ from the premises in $\Gamma \cup \{P\}$, then

$$\dfrac{\begin{array}{c} \Gamma, P^x \\ \mathscr{D} \\ \perp \end{array}}{\neg P}\ x$$

is a deduction tree for ¬P from the premises Γ. All premises P labeled x, are dis-charged.

The above rule can be viewed as a proof-
contradiction principle applied to
negated propositions.
Similarly, the ¬-elimination rule is a special case of

elimination applied to
$\neg P (= P \Rightarrow \perp)$ and P

Definition 1.4. The axioms and inference rules of the system $\mathcal{NG}_c^{\Rightarrow,\wedge,\vee,\perp}$

(of
propositional classical logic, Gentzen-sequent style) are listed below

$$\Gamma, x\colon P \to P \quad \text{(Axioms)}$$

$$\frac{\Gamma, x\colon P \to Q}{\Gamma \to P \Rightarrow Q} \quad (\Rightarrow\textit{-intro})$$

$$\frac{\Gamma \to P \Rightarrow Q \quad \Gamma \to P}{\Gamma \to Q} \quad (\Rightarrow\text{-}elim)$$

$$\frac{\Gamma \to P \quad \Gamma \to Q}{\Gamma \to P \wedge Q} \quad (\wedge\text{-}intro)$$

$$\frac{\Gamma \to P \wedge Q}{\Gamma \to P} \quad (\wedge\text{-}elim) \qquad \frac{\Gamma \to P \wedge Q}{\Gamma \to Q} \quad (\wedge\text{-}elim)$$

$$\frac{\Gamma \to P}{\Gamma \to P \vee Q} \quad (\vee\text{-}intro) \qquad \frac{\Gamma \to Q}{\Gamma \to P \vee Q} \quad (\vee\text{-}intro)$$

$$\frac{\Gamma \rightarrow P \vee Q \quad \Gamma, x: P \rightarrow R \quad \Gamma, y: Q \rightarrow R}{\Gamma \rightarrow R} \quad (\vee\text{-}elim)$$

$$\frac{\Gamma \to \bot}{\Gamma \to P} \quad (\bot\text{-}elim)$$

$$\frac{\Gamma, x : \neg P \to \bot}{\Gamma \to P} \quad (by\text{-}contra)$$

$$\frac{\Gamma, x : P \to \bot}{\Gamma \to \neg P} \quad (\neg\text{-introduction})$$

$$\frac{\Gamma \to \neg P \qquad \Gamma \to P}{\Gamma \to \bot} \quad (\neg\text{-elimination})$$

A deduction tree is either a one-node tree labeled with an axiom or a tree con-
structed using the above inference rules.

A proof tree is a deduction tree whose con-
clusion is a sequent with an empty set of premises (a sequent of the form $/ 0 \rightarrow P$)

The rule ($\perp$-elim) is trivial (does nothing) when P=$\perp$, therefore from now on we
assume that P $\neq \perp$. Propositional minimal logic, denoted $\mathcal{NG}_m^{\Rightarrow,\wedge,\vee,\perp}$

, is obtained by dropping the ($\perp$-elim) and (by-contra) rules. Propositional intuitionistic logic, denoted $\mathcal{NG}_i^{\Rightarrow,\wedge,\vee,\perp}$

, is obtained by dropping the (by-contra) rule a proposition P is provable from Γ, we mean that we can construct a proof tree whose conclusion is P and whose set of premises is Γ, in one of the systems

$$\mathcal{N}_c^{\Rightarrow,\wedge,\vee,\perp} \text{ or } \mathcal{N}\mathcal{G}_c^{\Rightarrow,\wedge,\vee,\perp}$$

When P is provable from Γ, most people write $\Gamma \vdash P$, or $\vdash \Gamma \rightarrow P$, sometimes with the name of the corresponding proof system tagged as a subscript on the sign $\vdash$ if necessary to avoid ambiguities. When Γ is empty, we just say P is provable (provable in intuitionistic logic, and so on) and write $\vdash P$

We treat logical equivalence as a derived connective: that is, we view $P \equiv Q$ as an abbreviation for $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. In view of the inference rules for $\wedge$, we see that to prove a logical equivalence $P \equiv Q$, we just have to prove both implications $P \Rightarrow Q$ and $Q \Rightarrow P$

Could we Interpret $\neg P$ as “P is not provable.” ?

Indeed, if $\neg P$ and P were both provable, then $\bot$ would be provable. So, P should not be provable if $\neg P$ is. However, if P is not provable, then $\neg P$ is not provable in general. There are plenty of propositions such that neither P nor $\neg P$ is provable (for instance, P, with P an atomic proposition).
Thus, the fact that P is not provable is not equivalent to the provability of $\neg P$ and we should not interpret $\neg P$ as “P is not provable.”

For example, if remain1(n) is the proposition that asserts n is a whole number of the form 4k+1 and remain3(n) is the proposition
that asserts n is a whole number of the form 4k+3 (for some whole number k), then
we can prove the implication
$(\text{remain1}(n) \lor \text{remain3}(n)) \Rightarrow \text{odd}(n)$

The proof-by-contradiction rule formalizes the method of proof by contradiction.
That is, in order to prove that P can be deduced from some premises Γ, one may
assume the negation ¬P of P (intuitively, assume that P is false) and then derive a
contradiction from Γ and ¬P (i.e., derive falsity). Then, P actually follows from Γ
without using ¬P as a premise, that is, ¬P is discharged. For example, let us prove
by contradiction that if n^2 is odd, then n itself must be odd, where n is a natural
number

Proposition 1.1. The proposition P ∨ ¬P is provable in classical logic

Proof. We prove that $P \vee (P \Rightarrow \perp)$ is provable by using the proof-by-contradiction rule as shown below:

$$
\dfrac{
  ((P \vee (P \Rightarrow \perp)) \Rightarrow \perp)^{y}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        ((P \vee (P \Rightarrow \perp)) \Rightarrow \perp)^{y}
        \qquad
        \dfrac{P^{x}}{P \vee (P \Rightarrow \perp)}
      }{\perp}
    }{P \Rightarrow \perp}\, x
  }{P \vee (P \Rightarrow \perp)}
}{
  \dfrac{\perp}{P \vee (P \Rightarrow \perp)}\; y \text{ (by-contra)}
}
$$

□