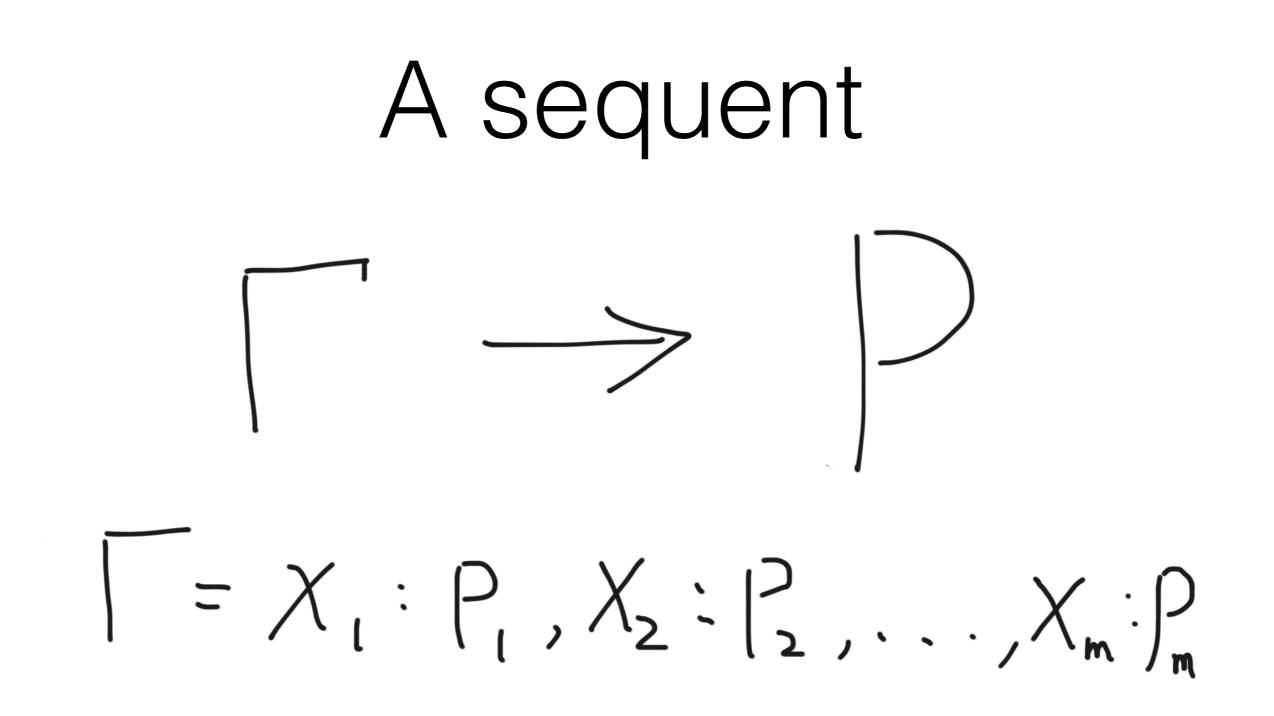
Gentzen's deduction system

We use variables for the labels, and a packet labeled with x consisting of occurrences of the proposition P is written as x: P.

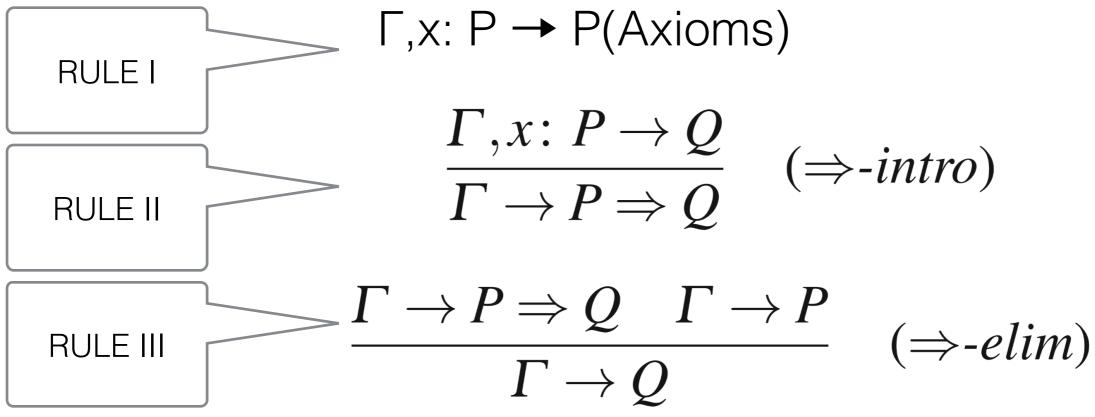
Thus, in a sequent $\Gamma \rightarrow P$, the expression Γ is any finite set of the for m x1: P1,...,xm: Pm,where the xi are pairwise distinct (but the Pi need not be dis-tinct). Given $\Gamma = x1$: P1,...,xm: Pm, the notation Γ ,x: P is only well defined when $x \rightleftharpoons xi$ for all i, $1 \le i \le m$, in which case it denotes the set x1: P1,...,xm: Pm,x: P.



Definition 1.2. The axioms and inference rules of the system \mathcal{NG}

(implicational

logic, Gentzen-sequent style (the G in N G stands for Gentzen)) are listed below:



$$\begin{array}{c|c} x: P, y: Q \to P \\ \hline x: P \to Q \Rightarrow P \\ \hline \to P \Rightarrow (Q \Rightarrow P) \end{array}$$

$$\frac{(A \Rightarrow (B \Rightarrow C))^{z} \qquad A^{x}}{B \Rightarrow C} \qquad \frac{(A \Rightarrow B)^{y} \qquad A^{x}}{B}$$
$$\frac{(A \Rightarrow B)^{y}}{A^{x}} \qquad \frac{A^{x}}{B}$$
$$\frac{C}{A \Rightarrow C} \qquad y$$
$$\frac{(A \Rightarrow C)^{y}}{(A \Rightarrow B) \Rightarrow (A \Rightarrow C)} \qquad y$$
$$\frac{(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))}{(A \Rightarrow C) \Rightarrow (A \Rightarrow C))} \qquad z$$

$$\Gamma = x : A \Rightarrow (B \Rightarrow C), y : A \Rightarrow B, z : A.$$

$$\frac{\Gamma \to A \Rightarrow (B \Rightarrow C) \qquad \Gamma \to A}{\Gamma \to B \Rightarrow C} \qquad \frac{\Gamma \to A \Rightarrow B \qquad \Gamma \to A}{\Gamma \to B}$$

$$\frac{x : A \Rightarrow (B \Rightarrow C), y : A \Rightarrow B, z : A \to C}{x : A \Rightarrow (B \Rightarrow C), y : A \Rightarrow B \to A \Rightarrow C}$$

$$\frac{x : A \Rightarrow (B \Rightarrow C), y : A \Rightarrow B \to A \Rightarrow C}{x : A \Rightarrow (B \Rightarrow C) \to (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}$$

$$\rightarrow (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$$

$$\Gamma = x: A \Rightarrow (B \Rightarrow C), y: A \Rightarrow B, z: A.$$

$$\frac{\Gamma \to A \Rightarrow (B \Rightarrow C) \qquad \Gamma \to A}{\Gamma \to B \Rightarrow C} \qquad \frac{\Gamma \to A \Rightarrow B \qquad \Gamma \to A}{\Gamma \to B}$$

$$\frac{x: A \Rightarrow (B \Rightarrow C), y: A \Rightarrow B, z: A \to C}{x: A \Rightarrow (B \Rightarrow C), y: A \Rightarrow B, z: A \to C}$$

$$\frac{x: A \Rightarrow (B \Rightarrow C), y: A \Rightarrow B \to A \Rightarrow C}{x: A \Rightarrow (B \Rightarrow C), y: A \Rightarrow B \to A \Rightarrow C}$$

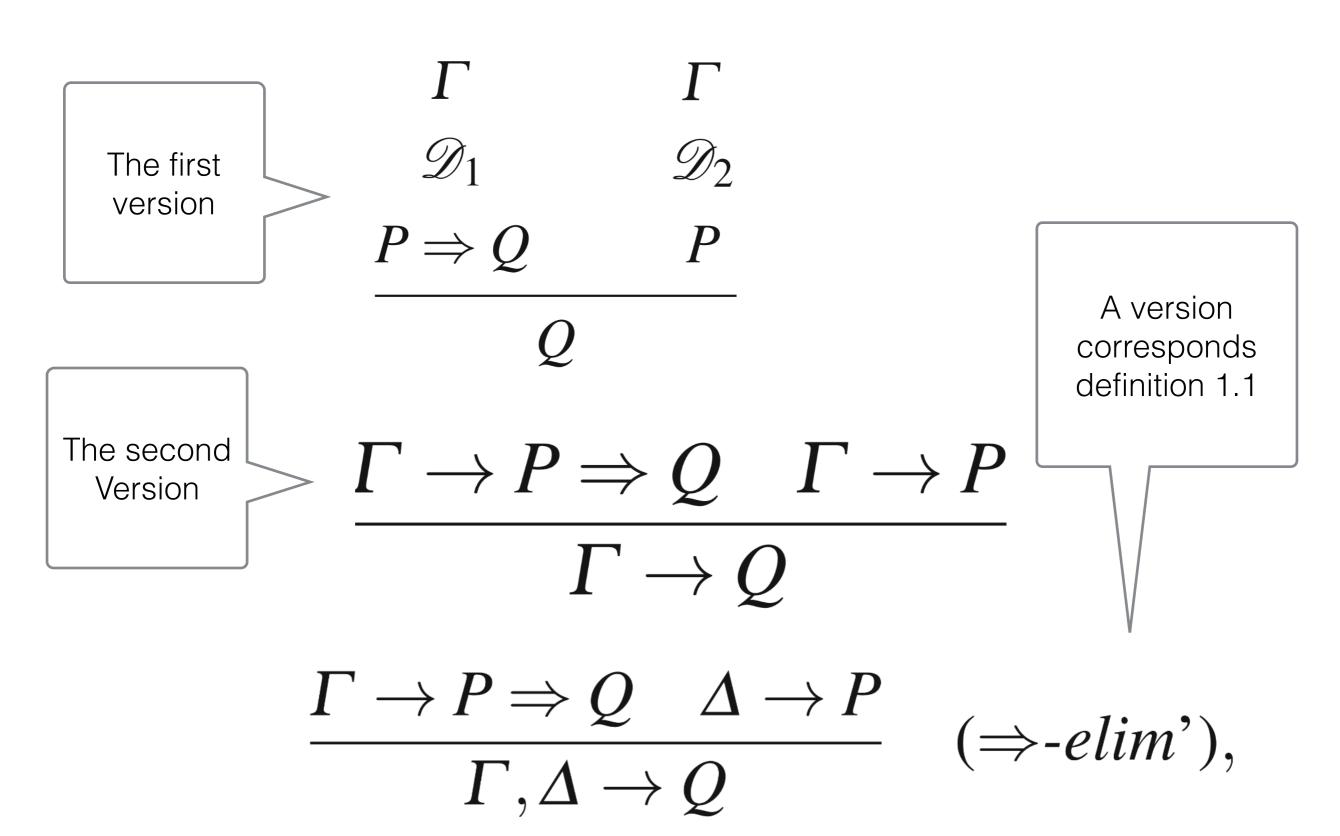
$$\frac{x: A \Rightarrow (B \Rightarrow C) \to (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}{\to (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))}$$

$$\frac{(A \Rightarrow (B \Rightarrow C))^{z} \qquad A^{x}}{B \Rightarrow C} \qquad (A \Rightarrow B)^{y} \qquad A^{x}}{B}$$

$$\frac{\frac{C}{A \Rightarrow C}}{(A \Rightarrow B) \Rightarrow (A \Rightarrow C)} \qquad y$$

$$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$$

Elimination rule



- Definition 1.3. The axioms, inference rules, and deduction trees for (propositional) classical logic are defined as follows.
- Axioms:
- (i) Every one-node tree labeled with a single proposition P is a deduction tree for P with set of premises {P}

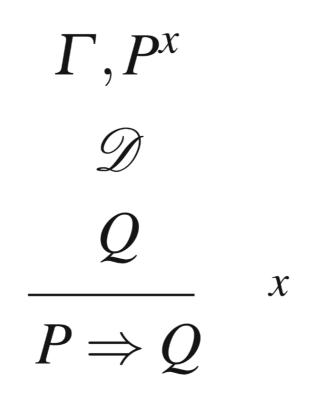
(ii) The tree Γ, P P

is a deduction tree for P with multiset of premises $\Gamma \cup \{P\}$.

The \Rightarrow -introduction rule:

If D is a deduction of Q from the premises in $\Gamma \cup \{P\}$, the is a deduction tree for P \Rightarrow Q from Γ . All premises P

labeled x are discharged



The \Rightarrow -elimination rule (or modus ponens):

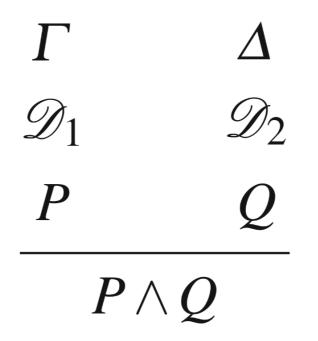
If D1 is a deduction tree for $P \Rightarrow Q$ from the

premises Γ , and D2. is a deduction for P from the premises Δ , the

 $\begin{array}{ccc}
\Gamma & \Delta \\
\widehat{\mathscr{D}}_1 & \widehat{\mathscr{D}}_2 \\
P \Rightarrow Q & P \\
\hline
Q
\end{array}$

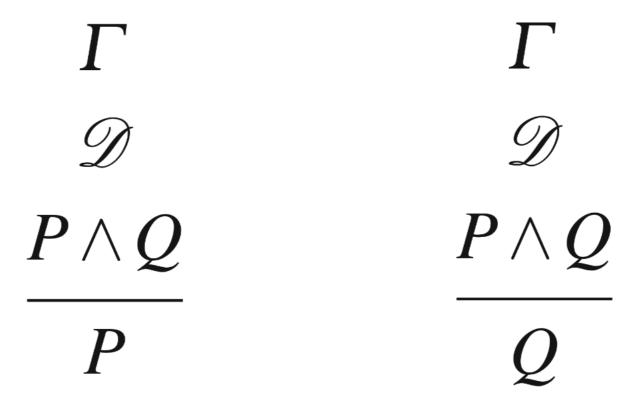
is a deduction tree for Q from the premises in $\Gamma ~ U \, \Delta$

The Λ -introduction rule: If D1 is a deduction tree for P from the premises Γ , and D2is deduction for Q from the premises Δ , then



is a deduction tree for $\mathsf{P}\land\mathsf{Q}$ from the premises in Γ $\cup\Delta$

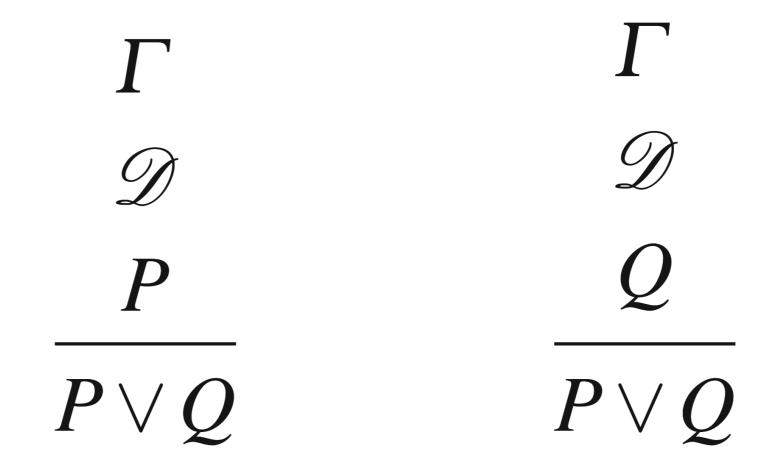
The Λ -elimination rule: If D is a deduction tree for $P \wedge Q$ from the premises Γ , then



are deduction trees for P and Q from the premises Γ

The V-introduction rule:

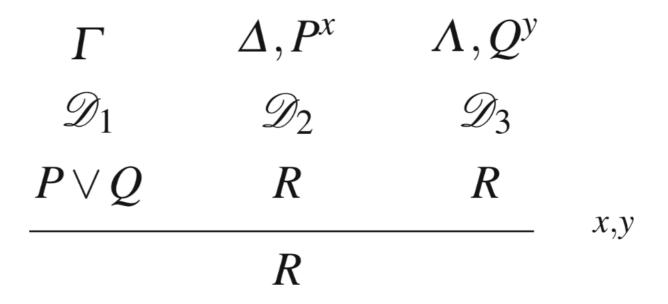
If D is a deduction tree for P or for Q from the premises Γ , then



are deduction trees for PVQ from the premises in Γ

The \Rightarrow -elimination rule (or modus ponens):

If D1 is a deduction tree for P \Rightarrow Q from the premises Γ , and D2 is a deduction for P from the premises Δ , then



is a deduction tree for Q from the premises in Γ U Δ U Λ . All premises P labeled x and all premises Q labeled y are discharged

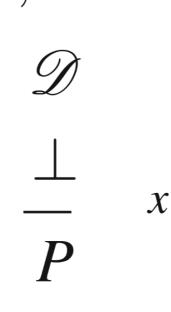
The \perp -elimination rule:

If D is a deduction tree for \perp from the premises Γ , then

 $\frac{\Gamma}{\mathscr{D}} \\
\frac{\bot}{P}$

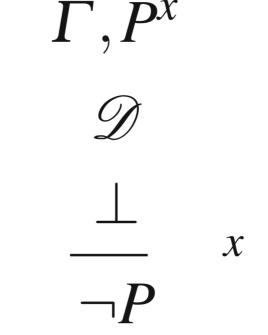
is a deduction tree for P from the premises Γ , for any proposition P

The proof-by-contradiction rule (also known as reductio ad absurdum rule, for short RAA): If D is a deduction tree for \perp from the premises in Γ $\cup \{\neg P\}$, then $\Gamma, \neg P^x$



is a deduction tree for P from the premises Γ. All premises ¬P labeled x, are discharged.

Because $\neg P$ is an abbreviation for $P \Rightarrow \bot$, the \neg -introduction rule is a special case of the \Rightarrow -introduction rule (with $Q = \bot$). The \neg -introduction rule: If D is a deduction tree for \bot from the premises in $\Gamma \cup \{P\}$, then ΓP^{x}



is a deduction tree for $\neg P$ from the premises Γ . All premises P labeled x, are dis-charged.

The above rule can be viewed as a proofcontradiction principle applied to negated propositions. Similarly, the \neg -elimination rule is a special case of elimination applied to $\neg P(=P \Rightarrow \bot)$ and P Definition 1.4. The axioms and inference rules of the system $\mathcal{NG}^{\Rightarrow}, \wedge, \vee, \perp$

(of

propositional classical logic, Gentzen-sequent style) are listed below

$$\Gamma, x: P \to P \quad \text{(Axioms)}$$
$$\frac{\Gamma, x: P \to Q}{\Gamma \to P \Rightarrow Q} \quad (\Rightarrow\text{-intro})$$

 $\frac{\Gamma \to P \Rightarrow Q \quad \Gamma \to P}{=} \quad (\Rightarrow -elim)$ $\Gamma \to Q$ $\Gamma \to P \quad \Gamma \to Q$ $(\wedge$ -intro) $\Gamma \to P \land Q$

$$\frac{\Gamma \to P \land Q}{\Gamma \to P} \quad (\land \text{-elim}) \qquad \frac{\Gamma \to P \land Q}{\Gamma \to Q} \quad (\land \text{-elim})$$

$$\frac{\Gamma \to P}{\Gamma \to P \lor Q} \quad (\lor \text{-intro}) \qquad \frac{\Gamma \to Q}{\Gamma \to P \lor Q} \quad (\lor \text{-intro})$$

$\frac{\Gamma \to P \lor Q \quad \Gamma, x \colon P \to R \quad \Gamma, y \colon Q \to R}{\Gamma \to R} \quad (\lor \text{-elim})$

$$\frac{\Gamma \to \bot}{\Gamma \to P} \quad (\bot \text{-elim})$$
$$\Upsilon, x: \neg P \to \bot$$

$$\frac{\Gamma, x: \neg P \to \bot}{\Gamma \to P} \quad (by\text{-contra})$$

$$\frac{\Gamma, x \colon P \to \bot}{\Gamma \to \neg P} \quad (\neg-\text{introduction})$$
$$\frac{\Gamma \to \neg P \quad \Gamma \to P}{\Gamma \to \bot} \quad (\neg-\text{elimination})$$

A deduction tree is either a one-node tree labeled with an axiom or a tree constructed using the above inference rules.

A proof tree is a deduction tree whose conclusion is a sequent with an empty set of premises (a sequent of the form / $0 \rightarrow P$) The rule (\perp -elim) is trivial (does nothing) when $P=\perp$, therefore from now on we assume that $P \neq \perp$. Propositional minimal logic, denoted $\mathcal{NG}_{m}^{\Rightarrow,\wedge,\vee,\perp}$

, is obtained by dropping the (\perp -elim) and (by-contra) rules. Propositional intuitionistic logic, denoted $\mathcal{NG}_i^{\Rightarrow,\wedge,\vee,\perp}$

, is obtained by dropping the (by-contra) rule

a proposition P is provable from Γ , we mean that we can construct a proof tree whose conclusion is P and whose set of premises is Γ , in one of the systems

 $\mathcal{N}_{c}^{\Rightarrow,\wedge,\vee,\perp}$ or $\mathcal{NG}_{c}^{\Rightarrow,\wedge,\vee,\perp}$

When P is provable from Γ , most people write $\Gamma \vdash P$, or $\vdash \Gamma \rightarrow P$, sometimes with the name of the corresponding proof system tagged as a subscript on the sign \vdash if necessary to avoid ambiguities. When Γ is empty, we just say P is provable (provable in intuitionistic logic, and so on) and write $\vdash P$ We treat logical equivalence as a derived connective: that is, we view $P \equiv Q$ as an abbreviation for $(P \Rightarrow Q) \land (Q \Rightarrow P)$. In view of the

inference rules for Λ , we see that to prove a logical equivalence $P \equiv Q$, we just have to prove both implications $P \Rightarrow Q$ and $Q \Rightarrow P$

Indeed, if $\neg P$ and P were both provable, then \perp would be provable. So, P should not be provable if $\neg P$ is. However, if P is not provable, then $\neg P$ is not provable in general. There are plenty of propositions such that neither P nor $\neg P$ is provable (for instance, P, with P an atomic proposition). Thus, the fact that P is not provable is not equivalent to the provability of $\neg P$ and we should not interpret $\neg P$ as "P is not provable."

For example, if remain1(n) is the proposition that asserts n is a whole number of the form 4k+1 and remain3(n) is the proposition that asserts n is a whole number of the form 4k+3(for some whole number k), then we can prove the implication (remain1(n) V remain3(n)) \Rightarrow odd(n) The proof-by-contradiction rule formalizes the method of proof by contradiction.

That is, in order to prove that P can be deduced from some premises Γ , one may assume the negation $\neg P$ of P (intuitively, assume that P is false) and then derive a contradiction from Γ and $\neg P$ (i.e., derive falsity). Then, P actually follows from Γ without using $\neg P$ as a premise, that is, $\neg P$ is discharged. For example, let us prove by contradiction that if n^2 is odd, then n itself must be odd, where n is a natural number

Proposition 1.1. The proposition PV ¬P is provable in classical logic

Proof. We prove that $P \lor (P \Rightarrow \bot)$ is provable by using the proof-by-contradiction rule as shown below:

