Let's Count!

Outline

- A Party
- Sets and the Like .
- The Number of Subsets
- The Approximate Number of Subsets.
- Sequences
- Permutations
- The Number of Ordered Subsets
- The Number of Subsets of a Given Size

Alice invites six guests to her birthday party: Bob, Carl, Diane, Eve, Frank, and George.

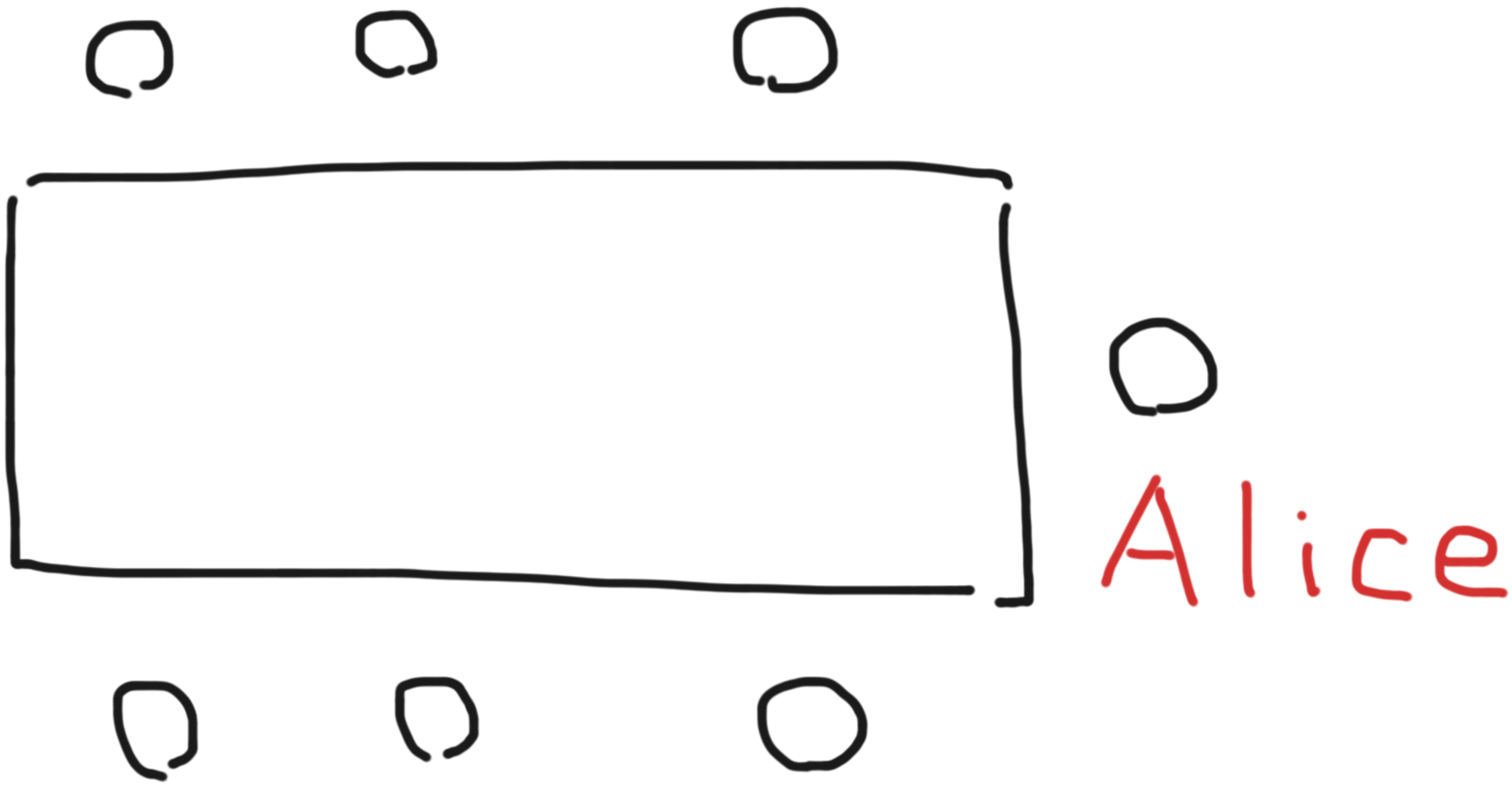
This group is strange anyway, because one of them asks,
"How many handshakes does this mean?"

"I shook 6 hands altogether," says Bob, "and 1 guess, so did everybody else."

"Since there are seven of us, this should mean 7 . 6 = 42 handshakes,"
ventures Carl.

"This seems too many" says Diane. "The same logic gives 2 handshakes
if two persons meet, which is clearly wrong."
"This is exactly the point: Every handshake was counted twice. We have
to divide 42 by 2 to get the right number: 21," with which Eve settles the
issue.

fill the seats



Alice suggests, "Let's change the seating every half hour, until we get every seating."

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How many ways can these people be seated at the table if Alice is fixed?

If they change seats every half hour, it will take 360 hours, that is, 15 days, to go through all the seating arrangements. Quite a party, at least as far as the duration goes!

After the cake, the crowd wants to dance (boys with girls, remember, this is a conservative European party). How many possible pairs can be formed?

(In the lottery they are talking about, 5 numbers are selected out of 90.)

"Let's pool our resources and win the lottery! All we have to do is to buy enough tickets so that no matter what they draw, we will have a ticket with the winning numbers. How many tickets do we need for this?"

"This is like the seating," says George. "Suppose we fill out the tickets so that Alice marks a number, then she passes the ticket to Bob, who marks a number and passes it to Carl, and so on. Alice has 90 choices, and no matter what she chooses, Bob has 89 choices, so there are 90 . 89 choices for the first two numbers, and going on similarly, we get 90 . 89 . 88 . 87 . 86 possible choices for the five numbers."

"Actually, I think this is more like the

"Actually, I think this is more like the handshake question," says Alice.
"If we fill out the tickets the way you suggested, we get the same ticket
more then once. For example, there will be a ticket where I mark 7 and
Bob marks 23, and another one where I mark 23 and Bob marks 7."

Carl jumps up: "Well, let's imagine a ticket, say, with numbers
7,23,31,34, and 55. How many ways do we get it? Alice could have marked
any of them; no matter which one it was that she marked, Bob could have
marked any of the remaining four. Now this is really like the seating prob-
lem. We get every ticket 5·4·3·2· 1 times."

"So," concludes Diane, "if we fill out the tickets the way George proposed,
then among the 90·89·88·87·86 tickets we get, every 5-tuple occurs notonly once, but 5 ·4· 3 . 2 . 1 times. So the number of different tickets is only

$$\frac{90 \times 89 \times 88 \times 87 \times 86}{1 \times 2 \times 3 \times 4 \times 5}$$

So they decide to play cards instead. Alice, Bob, Carl and Diane play bridge. Looking at his cards, Carl says, "I think I had the same hand last time."
"That is very unlikely" says Diane.
How unlikely is it? In other words, how many different hands can you have in bridge? (The deck has 52 cards, each player gets 13.) We hope you have noticed that this is essentially the same question as the lottery problem. Imagine that Carl picks up his cards one by one. The first card can be anyone of the 52 cards; whatever he picked up first, there are 51 possibilities for the second card, so there are 52 . 51 possibilities for the first two cards. Arguing similarly, we see that there are 52 . 51 . 50· . ·40 possibilities for the 13 cards.

There are 13 ·12· . ·2·1
orders in which he could have picked up his cards."

But this means that the number of different hands in bridge is

$$\frac{52 \cdot 51 \cdot 50 \cdots 40}{13 \cdot 12 \cdots 2 \cdot 1} = 635{,}013{,}559{,}600.$$

So the chance that Carl had the same hand twice in a row is one in 635,013,559,600, which is very small indeed.

Sets and the Like

Any collection of distinct objects, called elements, is a set.
The deck of cards is a set, whose elements are the cards. The participants in the party form a set, whose elements are Alice, Bob, Carl, Diane, Eve, Frank, and George (let us denote this set by P). Every lottery ticket of the type mentioned above contains a set of 5 numbers.

the
set of real numbers, denoted by IR; the set of rational numbers, denoted by
Q; the set of integers, denote by Z; the set of non-negative integers, denoted by Z+; the set of positive integers, denoted by N. The empty set, the set
with no elements, is another important (although not very interesting) set;
it is denoted by 0.

If A is a set and b is an element of A, we write

$$b \in A$$

The number of
elements of a set A (also called the cardinality of A) is denoted by |A|. Thus

$$|P| = 7, \quad |\emptyset| = 0, \quad |\mathbb{Z}| = \infty$$

p = {Alice, Bob, Carl, Diane, Eve, Frank, George}

$\{x \in Z : x > 0\}$

$\{x \in p : x \text{ is a girl}\}$

$= \{\text{Alice}, \text{Diane}, \text{Eve}\}$

A set A is called a subset of a set B if every element of A is also an element of B. In other words, A consists of certain elements of B. We can allow A to consist of all elements of B (in which case A = B) or none of them (in which case A = 0), and still consider it a subset. So the empty set is a subset of every set. The relation that A is a subset of B is denoted by

$$A \subseteq B$$

$$\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z}_+ \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

The notation A $\subset$ B means that A is a subset of B but not all of B.

$$0 \subset N \subset Z_{+} \subset Z \subset Q \subset R$$

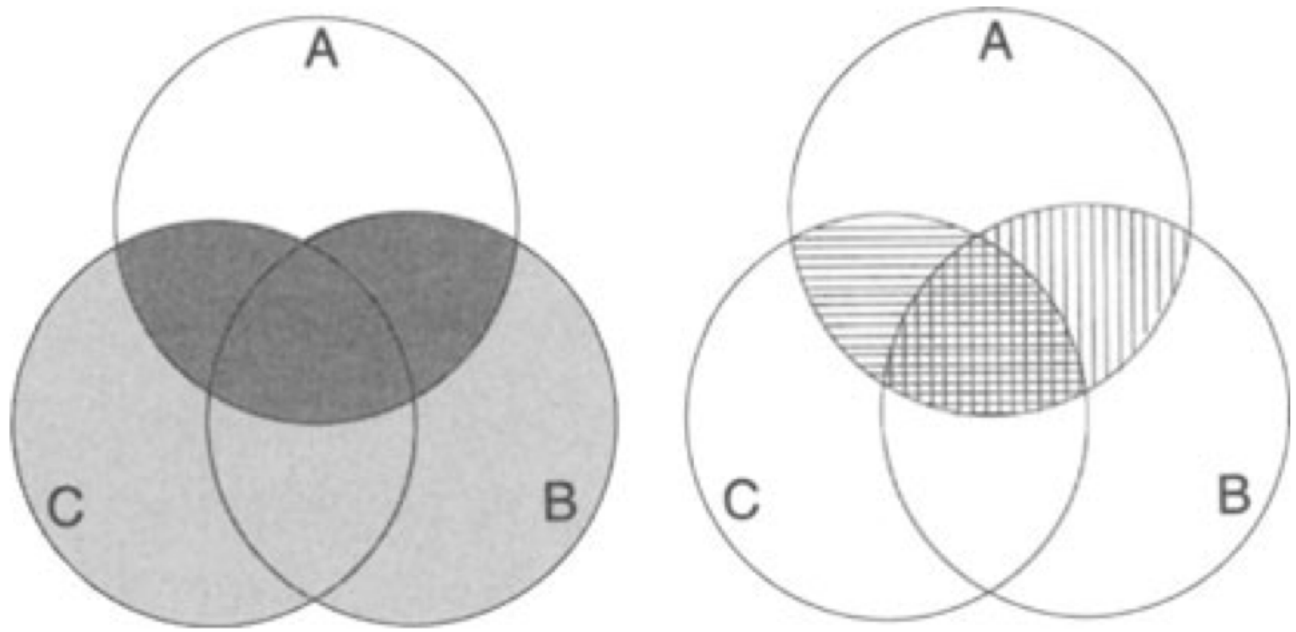
If we have two sets, we can define various other sets with their help. The intersection of two sets is the set consisting of those elements that are elements of both sets. The intersection of two sets A and B is denoted by A n B.

The union of two sets is the set consisting of those elements that are elements of at least one of the sets. The union of two sets A and B is denoted
by AUB.

The difference of two sets A and B is the set of elements that belong to A but not to B. The difference of two sets A and B is denoted by A \ B.

The symmetric difference of two sets A and B is the set of elements that belong to exactly one of A and B. The symmetric difference of two sets A and B is denoted by

$$A \triangle B$$



The Venn diagram of three sets, and the sets on both sides of (1.1)

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C$

Commutative and associative properties

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A,$$

$$(A \cup B) \cup C = A \cup (B \cup C), \qquad (A \cap B) \cap C = A \cap (B \cap C).$$

Distributive Law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

The Number of Subsets

What is the number of all subsets of a set with n elements?

subsets of a set {a, b, c} with 3 elements:
0,{a},{b},{c},{a,b},{b,c},{a,c},{a,b,c}.

Number of elements	0	1	2	3
Number of subsets	1	2	4	8

Theorem 1.3.1 *A set with n elements has 2^n subsets.*

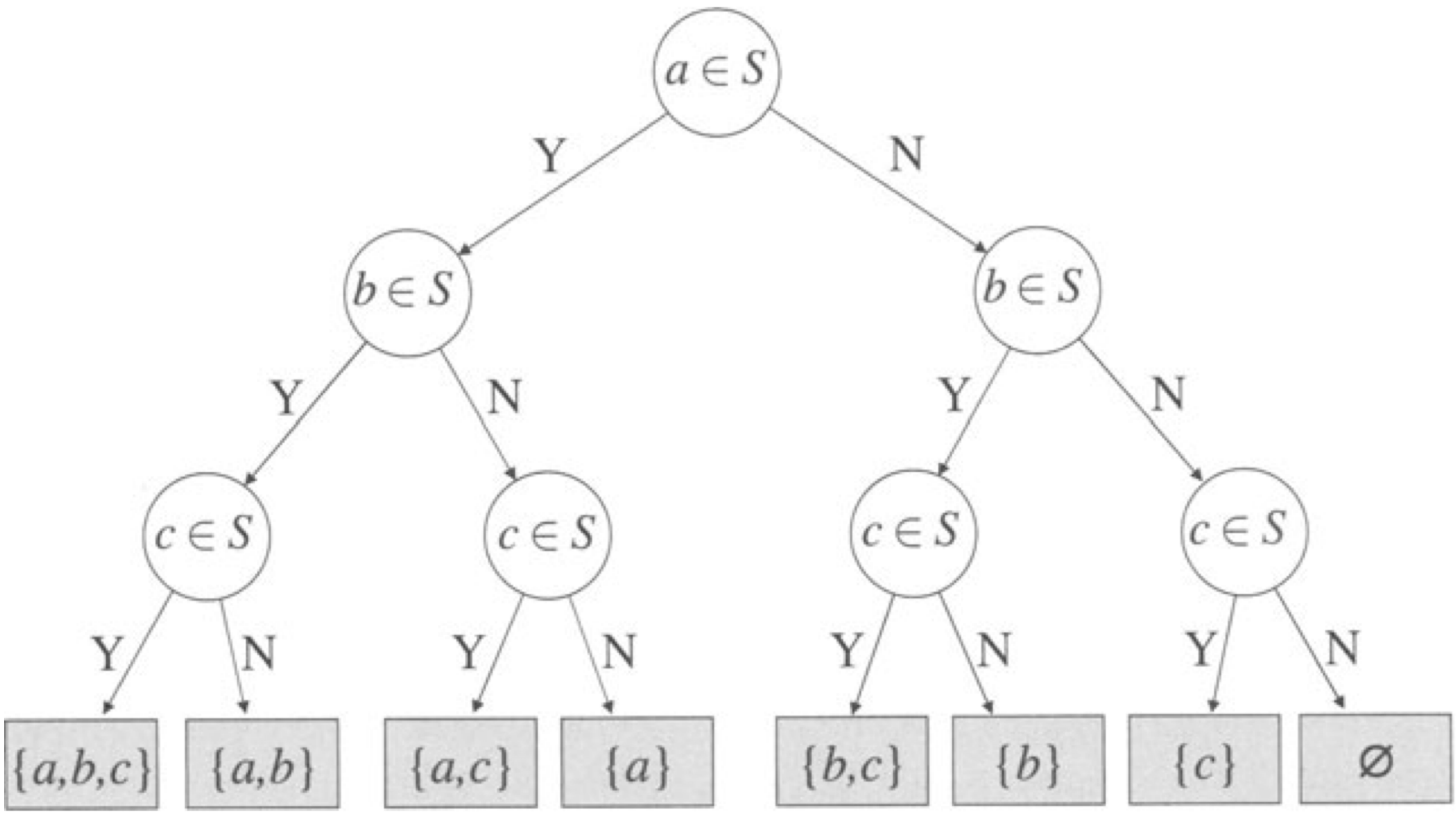


FIGURE 1.2. A decision tree for selecting a subset of $\{a, b, c\}$.

Binary Encoding

$$
\begin{array}{ccccccc}
0 & \Leftrightarrow & 0_2 & \Leftrightarrow & 000 & \Leftrightarrow & \emptyset \\
1 & \Leftrightarrow & 1_2 & \Leftrightarrow & 001 & \Leftrightarrow & \{c\} \\
2 & \Leftrightarrow & 10_2 & \Leftrightarrow & 010 & \Leftrightarrow & \{b\} \\
3 & \Leftrightarrow & 11_2 & \Leftrightarrow & 011 & \Leftrightarrow & \{b, c\} \\
4 & \Leftrightarrow & 100_2 & \Leftrightarrow & 100 & \Leftrightarrow & \{a\} \\
5 & \Leftrightarrow & 101_2 & \Leftrightarrow & 101 & \Leftrightarrow & \{a, c\} \\
6 & \Leftrightarrow & 110_2 & \Leftrightarrow & 110 & \Leftrightarrow & \{a, b\} \\
7 & \Leftrightarrow & 111_2 & \Leftrightarrow & 111 & \Leftrightarrow & \{a, b, c\}
\end{array}
$$

for every subset, we had exactly one corresponding number, and -for every number, we had exactly one corresponding subset.

A correspondence with these properties is called a one-to-one correspon-
dence (or bijection). If we can make a one-to-one correspondence between
the elements of two sets, then they have the same number of elements.