Let's Count!

Outline

- A Party
- Sets and the Like .
- The Number of Subsets
- The Approximate Number of Subsets.
- Sequences
- Permutations
- The Number of Ordered Subsets
- The Number of Subsets of a Given Size

Alice invites six guests to her birthday party: Bob, Carl, Diane, Eve, Frank, and George.

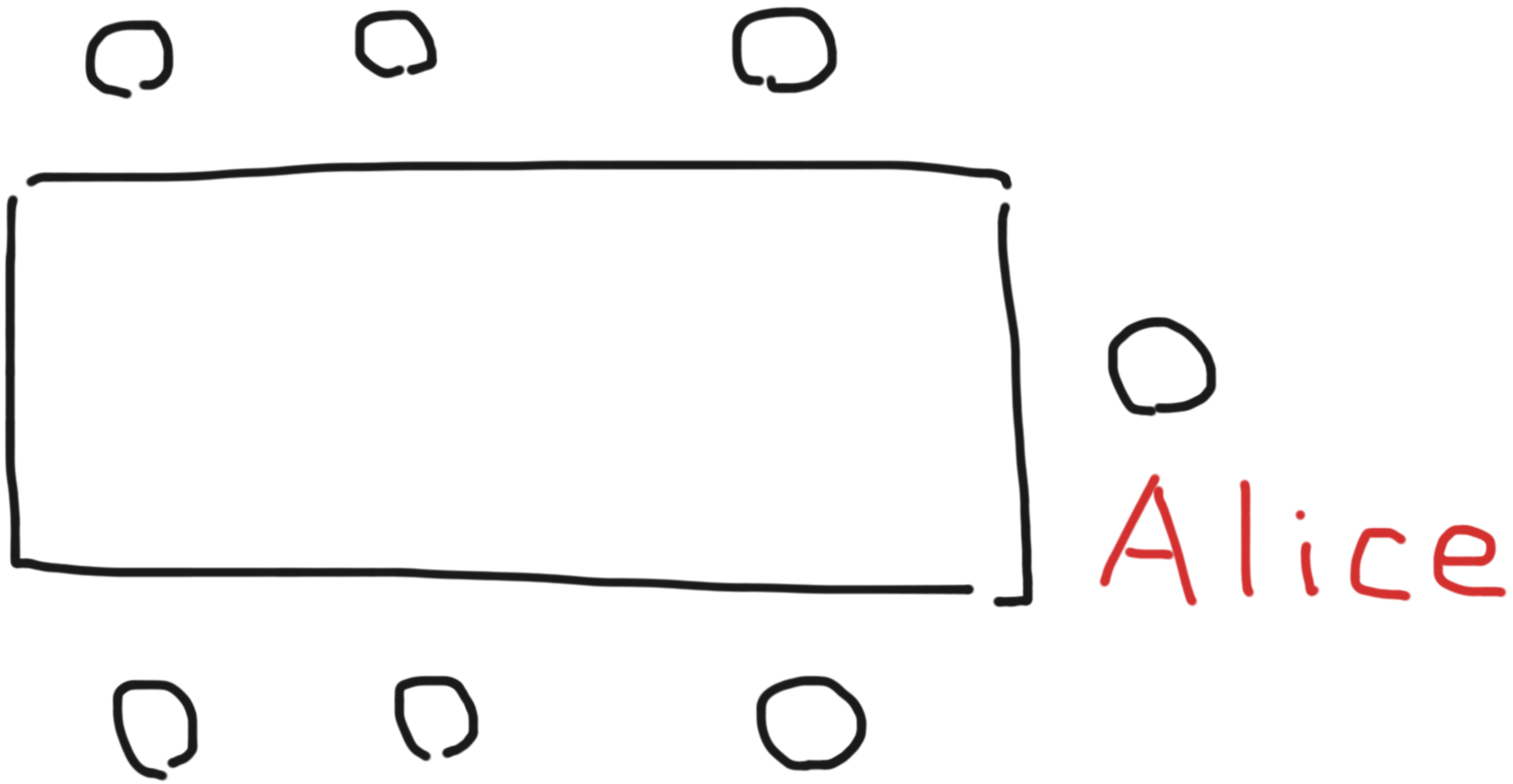
This group is strange anyway, because one of them asks,
"How many handshakes does this mean?"

"I shook 6 hands altogether," says Bob, "and 1 guess, so did everybody else."

"Since there are seven of us, this should mean $7 . 6 = 42$ handshakes,"
ventures Carl.

"This seems too many" says Diane. "The same logic gives 2 handshakes
if two persons meet, which is clearly wrong."
"This is exactly the point: Every handshake was counted twice. We have to divide 42 by 2 to get the right number: 21," with which Eve settles the issue.

fill the seats



- Alice suggests, "Let's change the seating every half hour, until we get every seating."

How many ways can these people be seated at the table if Alice is fixed?

If they change seats every half hour, it will take 360 hours, that is, 15 days, to go through all the seating arrangements. Quite a party, at least as far as the duration goes!

After the cake, the crowd wants to dance (boys with girls, remember, this is a conservative European party). How many possible pairs can be formed?

(In the lottery they are talking about, 5 numbers are selected out of 90.)

"Let's pool our resources and win the lottery! All we have to do is to buy enough tickets so that no matter what they draw, we will have a ticket with the winning numbers. How many tickets do we need for this?"

"This is like the seating," says George. "Suppose we fill out the tickets so that Alice marks a number, then she passes the ticket to Bob, who marks a number and passes it to Carl, and so on. Alice has 90 choices, and no matter what she chooses, Bob has 89 choices, so there are 90 . 89 choices for the first two numbers, and going on similarly, we get 90 . 89 . 88 . 87 . 86 possible choices for the five numbers."

"Actually, I think this is more like the handshake question," says Alice.
"If we fill out the tickets the way you suggested, we get the same ticket
more then once. For example, there will be a ticket where I mark 7 and
Bob marks 23, and another one where I mark 23 and Bob marks 7."

Carl jumps up: "Well, let's imagine a ticket, say, with numbers
7,23,31,34, and 55.

How many ways do we get it? Alice could have marked any of them; no matter which one it was that she marked, Bob could have marked any of the remaining four. Now this is really like the seating prob-
lem. We get every ticket $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ times."

"So," concludes Diane, "if we fill out the tickets the way George proposed,
then among the 90·89·88·87·86 tickets we get, every 5-tuple occurs notonly once, but 5 ·4· 3 . 2 . 1 times. So the number of different tickets is only

$$\frac{90 \times 89 \times 88 \times 87 \times 86}{1 \times 2 \times 3 \times 4 \times 5}$$

So they decide to play cards instead. Alice, Bob, Carl and Diane play bridge. Looking at his cards, Carl says, "I think I had the same hand last time."

"That is very unlikely" says Diane.

How unlikely is it? In other words, how many different hands can you have in bridge? (The deck has 52 cards, each player gets 13.) We hope you have noticed that this is essentially the same question as the lottery problem. Imagine that Carl picks up his cards one by one. The first card can be anyone of the 52 cards; whatever he picked up first, there are 51 possibilities for the second card, so there are $52 \cdot 51$ possibilities for the first two cards. Arguing similarly, we see that there are $52 \cdot 51 \cdot 50 \cdots 40$ possibilities for the 13 cards.

There are $13 \cdot 12 \cdot\ . \cdot 2 \cdot 1$
orders in which he could have picked up his cards."

But this means that the number of different hands in bridge is

$$\frac{52 \cdot 51 \cdot 50 \cdots 40}{13 \cdot 12 \cdots 2 \cdot 1} = 635{,}013{,}559{,}600.$$

So the chance that Carl had the same hand twice in a row is one in 635,013,559,600, which is very small indeed.

Sets and the Like

Any collection of distinct objects, called elements, is a set.
The deck of cards is a set, whose elements are the cards. The participants
in the party form a set, whose elements are Alice, Bob, Carl, Diane, Eve,
Frank, and George (let us denote this set by P). Every lottery ticket of the
type mentioned above contains a set of 5 numbers.

The set of real numbers, denoted by R; the set of rational numbers, denoted by Q; the set of integers, denote by Z; the set of non-negative integers, denoted by Z+; the set of positive integers, denoted by N. The empty set, the set with no elements, is another important (although not very interesting) set;
it is denoted by 0.

If A is a set and b is an element of A, we write

$$b \in A$$

The number of
elements of a set A (also called the cardinality of A) is denoted by IAI. Thus

$$|P| = 7, \quad |\emptyset| = 0, \quad |\mathbb{Z}| = \infty$$

p = {Alice, Bob, Carl, Diane, Eve, Frank, George}

$$\{x \in \mathbb{Z} : x > 0\}$$

$$\{x \in p : x \text{ is a girl}\}$$

$$= \{\text{Alice}, \text{Diane}, \text{Eve}\}$$

A set A is called a subset of a set B if every element of A is also an element of B. In other words, A consists of certain elements of B. We can allow A to consist of all elements of B (in which case A = B) or none of them (in which case A = 0), and still consider it a subset. So the empty set is a subset of every set. The relation that A is a subset of B is denoted by

$$A \subseteq B$$

$$0 \subseteq N \subseteq Z_+ \subseteq Z \subseteq Q \subseteq R$$

The notation A $\subset$ B means that A is a subset of B but not all of B.

$$0 \subset N \subset Z_+ \subset Z \subset Q \subset R$$

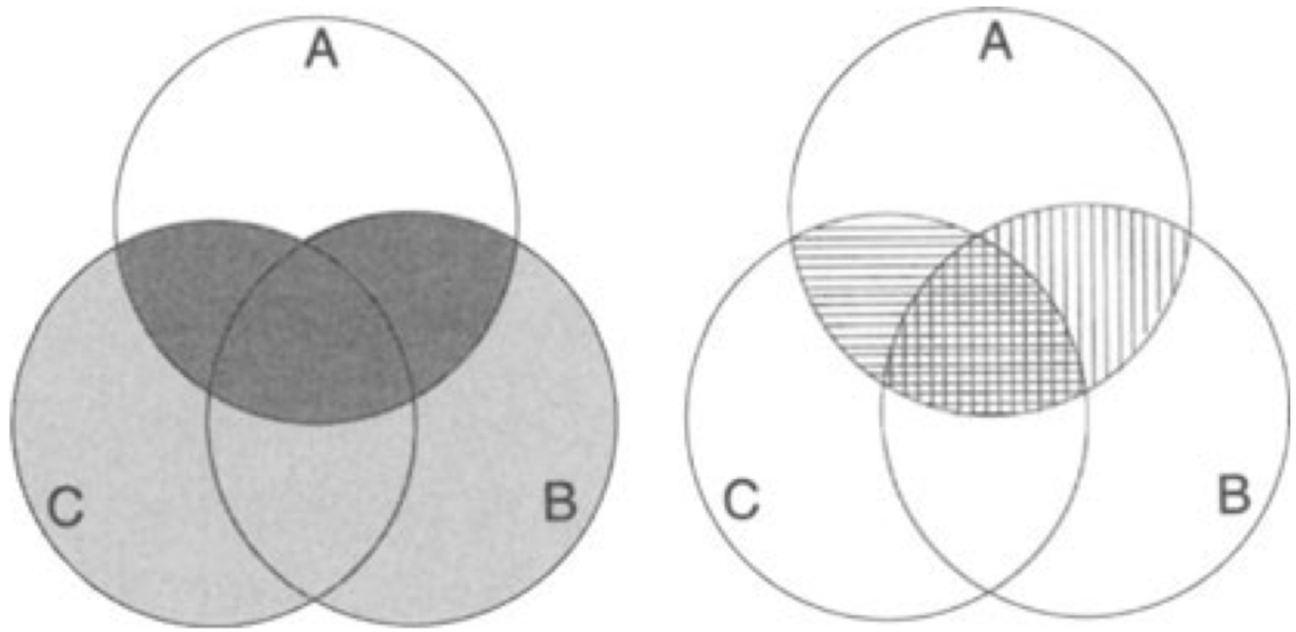
If we have two sets, we can define various other sets with their help. The intersection of two sets is the set consisting of those elements that are elements of both sets. The intersection of two sets A and B is denoted by A n B.

The union of two sets is the set consisting of those elements that are elements of at least one of the sets. The union of two sets A and B is denoted
by AUB.

The difference of two sets A and B is the set of elements that belong to A but not to B. The difference of two sets A and B is denoted by A \ B.

The symmetric difference of two sets A and B is the set of elements that belong to exactly one of A and B. The symmetric difference of two sets A and B is denoted by

$$A \triangle B$$



The Venn diagram of three sets, and the sets on both sides of (1.1)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C$$

Commutative and associative properties

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A,$$

$$(A \cup B) \cup C = A \cup (B \cup C), \qquad (A \cap B) \cap C = A \cap (B \cap C).$$

Distributive Law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

The Number of Subsets

What is the number of all subsets of a set with n elements?

subsets of a set {a, b, c} with 3 elements:
0,{a},{b},{c},{a,b},{b,c},{a,c},{a,b,c}.

Number of elements	0	1	2	3
Number of subsets	1	2	4	8

Theorem 1.3.1 *A set with n elements has 2^n subsets.*

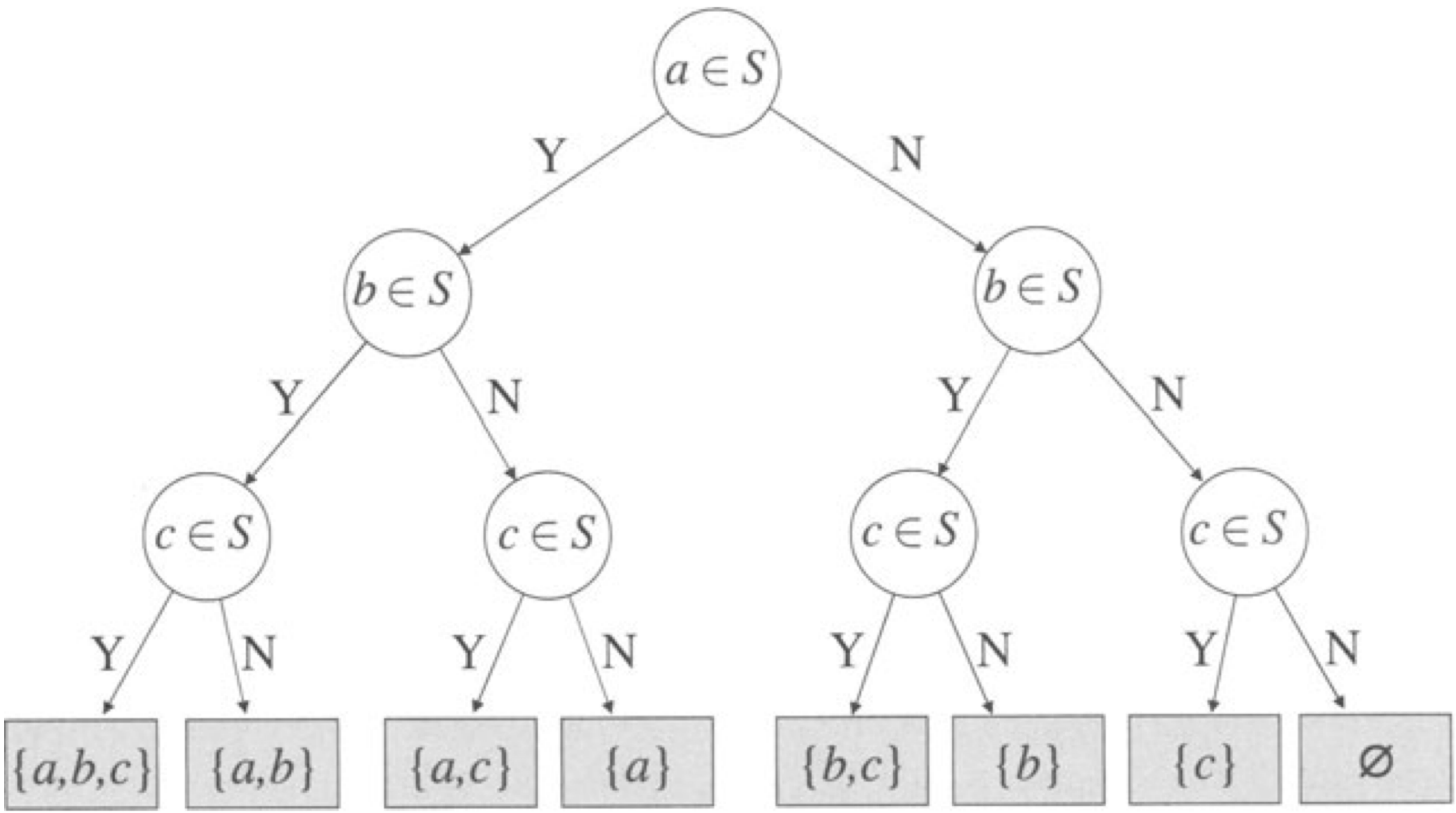


FIGURE 1.2. A decision tree for selecting a subset of $\{a, b, c\}$.

Binary Encoding

$$
\begin{array}{ccccccc}
0 & \Leftrightarrow & 0_2 & \Leftrightarrow & 000 & \Leftrightarrow & \emptyset \\
1 & \Leftrightarrow & 1_2 & \Leftrightarrow & 001 & \Leftrightarrow & \{c\} \\
2 & \Leftrightarrow & 10_2 & \Leftrightarrow & 010 & \Leftrightarrow & \{b\} \\
3 & \Leftrightarrow & 11_2 & \Leftrightarrow & 011 & \Leftrightarrow & \{b, c\} \\
4 & \Leftrightarrow & 100_2 & \Leftrightarrow & 100 & \Leftrightarrow & \{a\} \\
5 & \Leftrightarrow & 101_2 & \Leftrightarrow & 101 & \Leftrightarrow & \{a, c\} \\
6 & \Leftrightarrow & 110_2 & \Leftrightarrow & 110 & \Leftrightarrow & \{a, b\} \\
7 & \Leftrightarrow & 111_2 & \Leftrightarrow & 111 & \Leftrightarrow & \{a, b, c\}
\end{array}
$$

for every subset, we had exactly one corresponding number, and for every number, we had exactly one corresponding subset.

A correspondence with these properties is called a one-to-one correspondence (or bijection). If we can make a one-to-one correspondence between the elements of two sets, then they have the same number of elements.

The Approximate Number of Subsets

- 2^100 =
- 1267650600228229401496703205376

$$x = \log_{10} 2^{100}$$

$$10^{k-1} \le 2^{100} < 10^{k}$$

$$k-1 \le x = 100\log_{10} 2 < k$$

$$k-1 \le x = 30.1030 < k$$

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>> 100*(log10(2))

ans =

   30.1030

>>
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$K = 31$

$10^{30} \leq 2^{100}$

2^{100} at least has 30 digits

Theorem 1.5.1 The number of strings of length n composed of k given elements is k^n

Suppose that a database has 4 fields: the first, containing an 8-character abbreviation of an employee's name; the second, M or F for sex; the third, the birthday
of the employee, in the format mm-dd-yy (disregarding the problem of not being able to distinguish employees born in 1880 from employees born in
1980); and the fourth, a job code that can be one of 13 possibilities. How many different records are possible?

$$26^8 \cdot$$

$$2 \cdot 12 \cdot 31 \cdot 100 \cdot 13 = 201{,}977{,}536{,}857{,}907{,}200.$$

Theorem 1.5.2 Suppose that we want to form strings of length n by using any of a given set of kl symbols as the first element of the string, any of a given set of k2 symbols as the second element of the string, etc., any of a given set of kn symbols as the last element of the string. Then the total number of strings we can form is kl . k2 ... kn.

Permutations

If we have a list of n objects (an ordered set, where it is specified which element is the first, second, etc.), and we rearrange them so that they are in another order, this is called permuting them; the new order is called a permutation of the objects. We also call the rearrangement that does not change anything a permutation (somewhat in the spirit of calling the empty set a set). For example, the set {a, b, c} has the following 6 permutations:

abc,acb,bac,bca,cab,cba.

- Theorem 1.6.1 The number of permutations of n objects is n!.

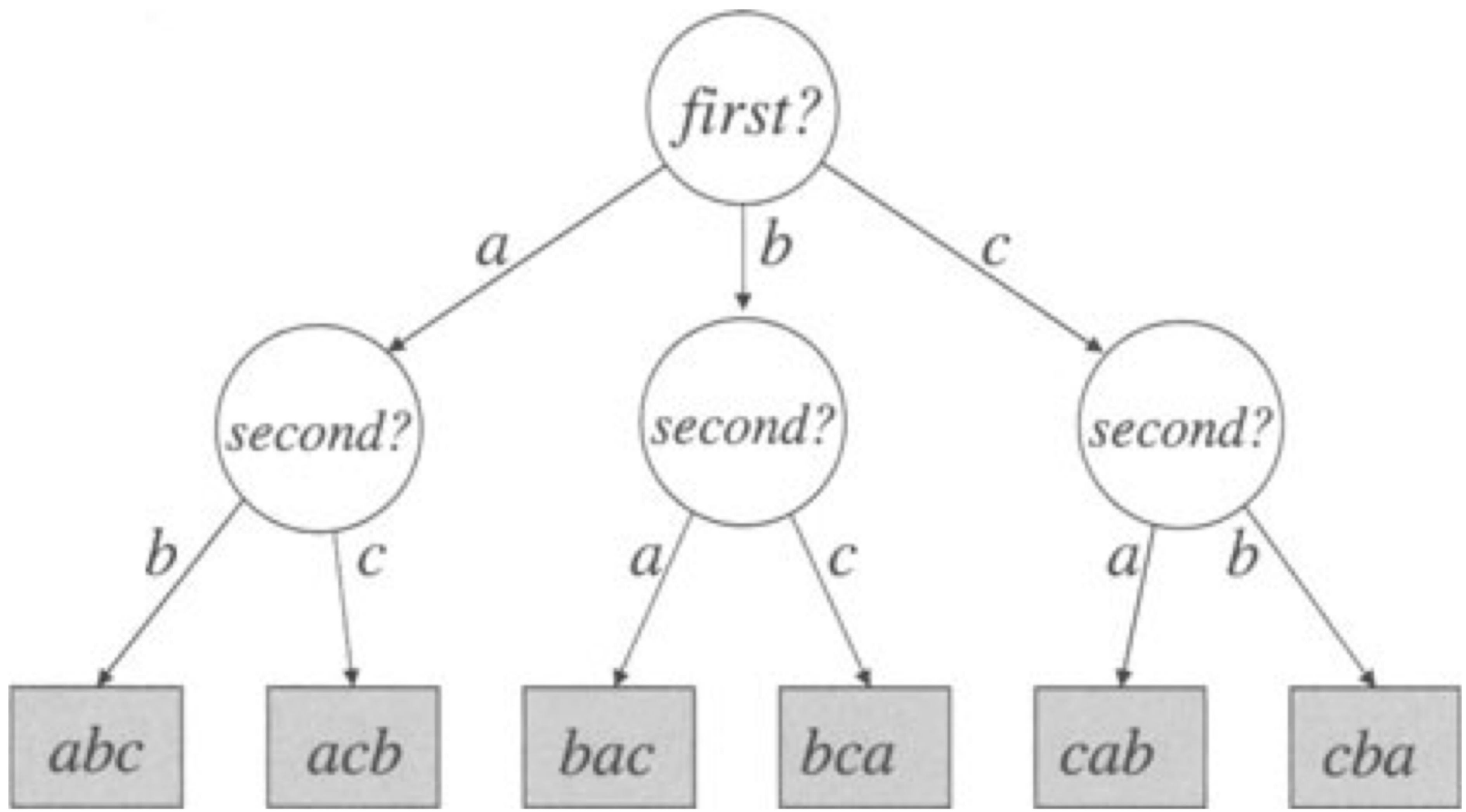

FIGURE 1.3. A decision tree for selecting a permutation of $\{a, b, c\}$.

- It is clear that for a set with n elements, n arrows leave the top node, and hence there are n nodes on the next level. Then n -1 arrows leave each of these, hence there are n(n -1) nodes on the third level. Then n -2 arrows leave each of these, etc. The bottom level has n! nodes. This shows that there are exactly n! permutations.

The Number of Ordered Subsets

- Theorem 1.7.1 The number of ordered k-element subsets of a set with n elements is
- $n(n-1)\cdots(n-k+1)$.

The Number of Subsets of a Given Size

- Theorem 1.8.1 The number of k-subsets of an n-set is

$$\frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

- Proof. Recall that if we count ordered subsets, we get n(n -1) ... (n - k + 1) = n!/(n - k)!, by Theorem 1.7.1. Of course, if we want to know the number of unordered subsets, then we have overcounted; every subset was counted exactly k! times (with every possible ordering of its elements). So we have to divide this number by k! to get the number of subsets with k elements (without ordering).

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

- Theorem 1.8.2 Binomial coefficients satisfy the following identities:

$$\binom{n}{k} = \binom{n}{n-k};$$

If $n, k > 0$, *then*

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k};$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$