Mathematical Reasoning, Proof Principles, and Logic II

In words, the $\Rightarrow$-introduction rule says that in order to prove an implication $P \Rightarrow Q$ from a set of premises Γ, we assume that P has already been proved, add P to the premises in Γ, and then prove Q from Γ and P. Once this is done, the premise P is deleted

However, the business about discharging the premise P when we are through with our argument is a bit puzzling. Most people probably never carry out this “discharge step” con-sciously, but such a process does take place implicitly

It might help to view the action of proving an implication P⇒Q as the construc-tion of a program that converts a proof of P into a proof of Q. Then, if we supply a
proof of P as input to this program (the proof of P ⇒ Q), it will output a proof of Q.

functional point of view

So, if we don't give the right kind of input to this program, for example, a "wrong
proof" of P, we should not expect the program to return a proof of Q.

However, this does not say that the program is incorrect; the program was designed to do the right thing only if it is given the right kind of input.functional point of view
(also called constructive),

if we take the simplistic view that P and Q assume the truth values true and false, we should not be shocked that if we give as input the value false (for P), then the truth value of the whole implication $P \Rightarrow Q$ is true.

The program P⇒Q is designed to produce the output value true (for Q) if it is given the input value true (for P).

So, this program only goes wrong when, given the input true (for P), it returns the value false (for Q). In this erroneous case, P ⇒ Q should indeed receive the value false. However, in all other cases, the program works cor-rectly, even if it is given the wrong input (false for P)

P stands for the statement
"Our candidate for president wins in Pennsylvania"
and Q stands for
"Our candidate is elected president."
Then, $P \Rightarrow Q$, asserts that if our candidate for

president wins in Pennsylvania
then our candidate is elected president.

If $P \Rightarrow Q$ holds, then if indeed our candidate for president wins in Pennsylvania then for sure our candidate will win the presidential election.

However, if our candi-date does not win in Pennsylvania, we can’t predict what will happen. Our candidate
may still win the presidential election but he may not

If our candidate president does not win in Pennsylvania, our prediction is not proven false. In this case, the statement $P \Rightarrow Q$ should be regarded as holding,

though perhaps uninteresting

$odd(n)$

Natural number

$Q(n,a,b)$: $a^n + b^n$ is divisable by $a+b$

implication

$$odd(n) \Rightarrow Q(n,a,b)$$

$$n = 2k+1, \quad k = 0, 1, \ldots$$

$$a^n + b^n = a^{2k+1} + b^{2k+1}$$

$$= (a+b) \sum_{i=0}^{2k} (-1)^i a^{2k-i} b^i$$

$$\text{Odd}(n) \Rightarrow Q(n, a, b)$$

provable

Implication odd(n) ⇒ Q(n,a,b).

If n is not odd, then the implication odd(n) ⇒ Q(n,a,b) yields no information about the provablity of the statement Q(n,a,b), and that is fine.

Indeed, if n is even and n ≥ 2, then in general, a^n+b^n is not divisible by a+b, but this may happen for some special values of n, a, and b, for example: n = 2, a = 2, b = 2

1. Only the leaves of a deduction tree may be discharged. Interior nodes, including the root, are never discharged.

2. Once a set of leaves labeled with some premise P marked with the label x has been discharged, none of these leaves can be discharged again. So, each label (say x) can only be used once. This corresponds to the fact that some leaves of our deduction trees get “killed off” (discharged).

3. A proof is a deduction tree whose leaves are all discharged (Γ is empty). This corresponds to the philosophy that if a proposition has been proved, then the validity of the proof should not depend on any assumptions that are still active.
We may think of a deduction tree as an unfinished proof tree.

4. When constructing a proof tree, we have to be careful not to include (acciden-tally) extra premises that end up not being discharged. If this happens, we prob-ably made a mistake and the redundant premises should be deleted. On the other hand, if we have a proof tree, we can always add extra premises to the leaves and create a new proof tree from the previous one by discharging all the new premises.

5. Beware, when we deduce that an implication P⇒Q is provable, we do not prove that P and Q are provable; we only prove that if P is provable then Q is provable

$$\cfrac{\cfrac{P^x}{P}}{P \Rightarrow P}\, x$$

$$\frac{P^x}{P \Rightarrow P}\ x$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(Q \Rightarrow R)^y \qquad \dfrac{(P \Rightarrow Q)^z \qquad P^x}{Q}}{R}}{P \Rightarrow R}\;x}{(Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\;y}{(P \Rightarrow Q) \Rightarrow ((Q \Rightarrow R) \Rightarrow (P \Rightarrow R))}\;z$$

The tree below is a deduction tree, beause two of its leaves are labeled with the premises $P \Rightarrow Q$ and $Q \Rightarrow R$, that have

not been discharged yet. So, this tree represents a deduction of $P \Rightarrow R$ from the

set of premises $\Gamma = \{P \Rightarrow Q, Q \Rightarrow R\}$ but

it is not a proof tree because Γ is not empty. However, observe that the original premise P,
labeled x, has been discharged

$$\dfrac{Q \Rightarrow R \qquad \dfrac{P \Rightarrow Q \qquad P^x}{Q}}{\dfrac{R}{P \Rightarrow R}\, x}$$

The next tree was obtained from the previous one by applying the $\Rightarrow$-

introduction rule which triggered the discharge of the premise $Q \Rightarrow R$ labeled y,

which is no longer active. However, the premise $P \Rightarrow Q$ is still active (has not been

discharged yet), so the tree below is a deduction tree of $(Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$ from

the set of premises $\Gamma = \{P \Rightarrow Q\}$. It is not yet a

proof tree inasmuch as Γ is not empty

$$
\cfrac{\cfrac{\cfrac{(Q \Rightarrow R)^y \qquad \cfrac{P \Rightarrow Q \qquad P^x}{Q}}{R}}{P \Rightarrow R}\; x}{(Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\; y
$$

$$\dfrac{\dfrac{\dfrac{P^{x}, Q^{y}}{P}}{P \Rightarrow P}\;x}{Q \Rightarrow (P \Rightarrow P)}\;y$$

$$\dfrac{\dfrac{\dfrac{P^x, Q^y}{P}}{Q \Rightarrow P}\;y}{P \Rightarrow (Q \Rightarrow P)}\;x$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{P^u, P^v, P^y, Q^w, Q^x}{P}}{P \Rightarrow P}\ y}{Q \Rightarrow (P \Rightarrow P)}\ x}{Q \Rightarrow (Q \Rightarrow (P \Rightarrow P))}\ w}{P \Rightarrow (Q \Rightarrow (Q \Rightarrow (P \Rightarrow P)))}\ v}{P \Rightarrow (P \Rightarrow (Q \Rightarrow (Q \Rightarrow (P \Rightarrow P))))}\ u
$$

We use variables for the labels, and a packet labeled with x consisting of occurrences of the proposition P is written as x: P.

Thus, in a sequent $\Gamma \rightarrow P$, the expression Γ is any finite set of the for m x1: P1,...,xm: Pm,where the xi are pairwise distinct (but the Pi need not be dis-tinct). Given Γ = x1: P1,...,xm: Pm, the notation Γ,x: P is only well defined when x $\neq$ xi for all i, $1 \leq i \leq m$, in which case it denotes the set x1: P1,...,xm: Pm,x: P.

Definition 1.2. The axioms and inference rules of the system

$\mathcal{NG}_m^{\Rightarrow}$

(implicational
logic, Gentzen-sequent style (the G in N G stands for Gentzen)) are listed below:
Γ,x: P ➞ P(Axioms

$$\frac{\Gamma, x : P \to Q}{\Gamma \to P \Rightarrow Q} \quad (\Rightarrow\textit{-intro})$$

$$\frac{\Gamma \to P \Rightarrow Q \qquad \Gamma \to P}{\Gamma \to Q} \quad (\Rightarrow\textit{-elim})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(A\Rightarrow(B\Rightarrow C))^{z} \qquad A^{x}}{B\Rightarrow C} \qquad \dfrac{(A\Rightarrow B)^{y} \qquad A^{x}}{B}}{C}}{A\Rightarrow C}\;x}{(A\Rightarrow B)\Rightarrow(A\Rightarrow C)}\;y}{\big(A\Rightarrow(B\Rightarrow C)\big)\Rightarrow\big((A\Rightarrow B)\Rightarrow(A\Rightarrow C)\big)}\;z$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(A \Rightarrow (B \Rightarrow C))^z \qquad A^x}{B \Rightarrow C} \qquad \dfrac{(A \Rightarrow B)^y \qquad A^t}{B}}{C}\; x}{A \Rightarrow C}\; y}{(A \Rightarrow B) \Rightarrow (A \Rightarrow C)}\; z}{(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))}}{A \Rightarrow \Big((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)) \Big)}
$$