# An introduction to combinatorics　(IV-1)

## An Introduction to Combinatorics


1.1. Introduction . . . . . . . . . . . . . .

1.2. Enumeration . . . . . . . . . . . . . .

1.3. Combinatorics and Graph Theory .

1.4. Combinatorics and Number Theory

1.5. Combinatorics and Geometry . . . .

1.6. Combinatorics and Optimization . .

1.7. Sudoku Puzzles . . . . . . . . . . . .

1.8. Discussion . . . . . . . . . . . . . . .

# three principal themes

- Discrete Structures : Graphs, digraphs, networks, designs, posets, strings, patterns, distributions, coverings, and partitions.

- Enumeration:  Permutations, combinations, inclusion/exclusion, generating functions, recurrence relations, and counting.

- Algorithms and Optimization : Sorting, spanning trees, shortest paths, eulerian circuits, hamiltonian cycles, graph coloring, planarity testing, network flows, bipartite, matchings, and chain partition

# Enumeration

- Many basic problems in combinatorics involve counting the number of distributions of objects into cells—where we may or may not be able to distinguish between the objects and the same for the cells. Also, the cells may be arranged in patterns. Here are concrete examples.

- How many ways to give ten dollars to three children?

$$a_1 \geq a_2 \geq a_3$$

$$a_1 + a_2 + a_3 = 10$$

$$a_1, a_2, a_3 \in N \cup \{0\}$$

$$a_1 \geq a_2 \geq a_3$$

$$a_1 + a_2 + a_3 = 10$$

$$a_1, a_2, a_3 \in N$$

- Now suppose that Amanda has ten books, in fact the top 10 books from the New York Times best-seller list, and decides to give them to her children. How many ways can she do this? Again, we note that there is a hidden assumption—the ten books are all different

$B_1, B_2, \ldots, B_{10}$

$S_1, S_2, S_3$ pairwise disjoint

$S_1 \cup S_2 \cup S_3$

$= \{B_1, \ldots, B_{10}\}$

$B_1, B_2, \ldots, B_{10}$

$S_1, S_2, S_3$ pairwise disjoint

$S_1 \cup S_2 \cup S_3$ non-empty

$= \{B_1, \ldots, B_{10}\}$

How would we possibly answer these kinds of questions if ten was really ten thousand (OK, we're not talking about children any more!) and three was three thousand? Could you write the answer on a single page in a book

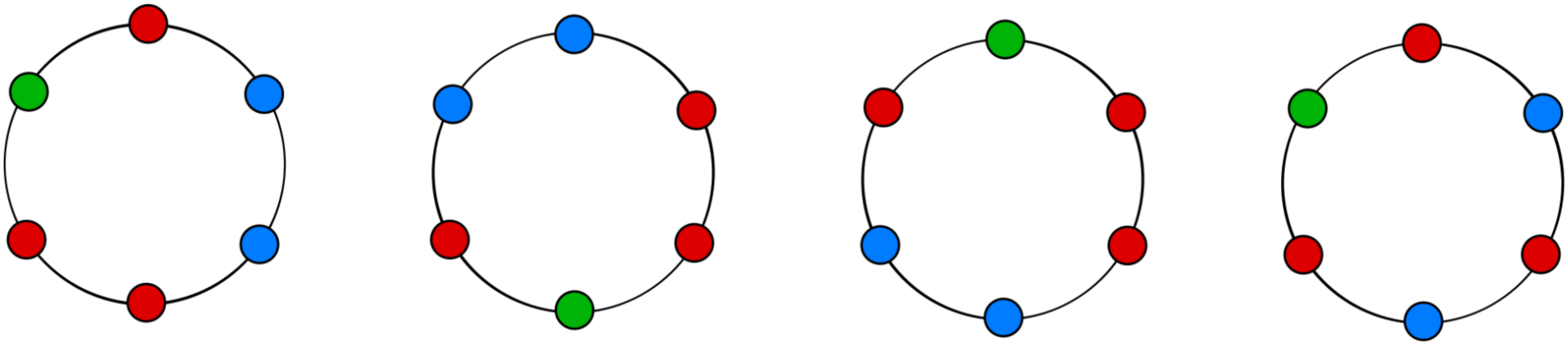


**Figure 1.1.:** Necklaces made with three colors

How many different necklaces of six beads can be formed using three reds, two blues and one green?

How many different necklaces of six beads can be formed using red, blue and green beads (not all colors have to be used)?

How many different necklaces of six beads can be formed using red, blue and green beads if all three colors have to be used?

How would we possibly answer these questions for necklaces of six thousand beads made with beads from three thousand different colors? What special software would be required to find the exact

l o n g

answer and how long would the computation take?

# Combinatorics and Graph Theory

A graph G consists of a vertex set V and a collection E of 2-element subsets of V. Elements of E are called edges. In our course, we will (almost always) use the convention that $V = \{1,2,3,\ldots,n\}$ for some positive integer n.

With this convention, graphs can be described precisely with a text file:

1. The first line of the file contains a single integer n, the number of vertices in the graph
2. Each of the remaining lines of the file contains a pair of distinct integers and specifies an edge of the graph

```
graph1.txt
9
6 2
1 5
1 7
6 8
9 1
4 3
5 7
1 3
5 9
7 9
```

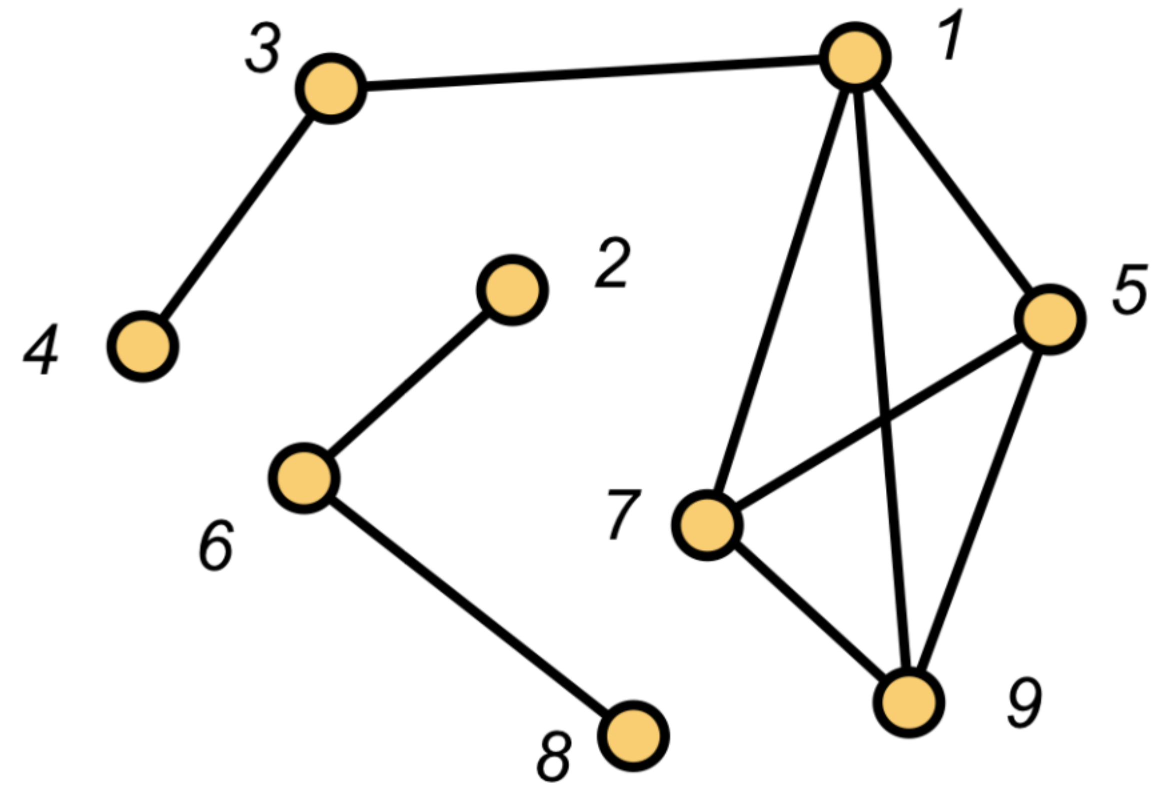



**Figure 1.2.:** A GRAPH DEFINED BY DA

G has 9 vertices and 10 edges.
2.{2,6}is an edge.
3. Vertices 5 and 9 are adjacent.
4.{5,4}is not an edge.
5. Vertices 3 and 7 are not adjacent.
6. P= (4,3,1,7,9,5)is a path of length 5 from vertex 4 to vertex 5.
7. C= (5,9,7,1)is cycle of length 4.
8. G is disconnected and has two components. One of the components has vertex set{2,6,8}.
9.{1,5,7}is a triangle.
10.{1,7,5,9}is a clique of size 4.
11.{4,2,8,5}is an independent set of size 4

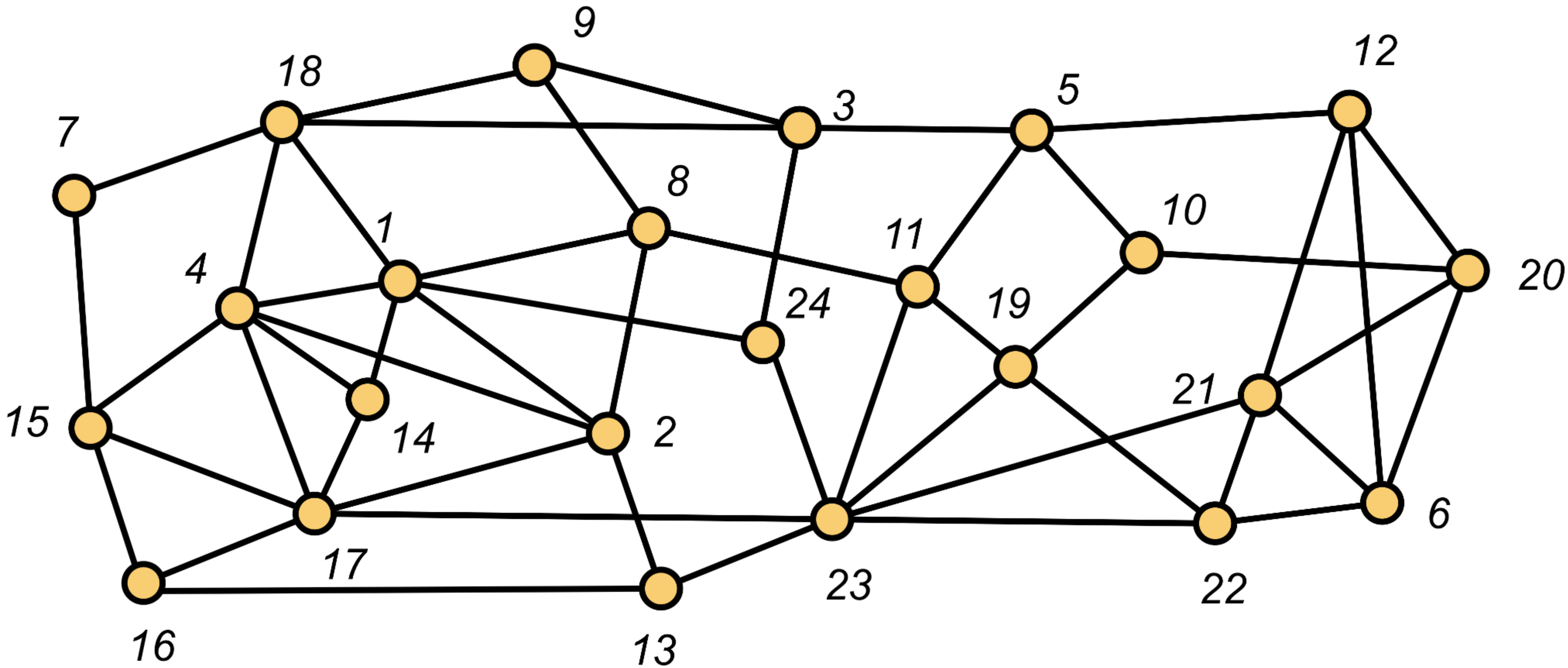



**Figure 1.3.:** A CONNECTED GRAPH

1. What is the largest k for which G has a path of length k?
2. What is the largest k for which G has a cycle of length k?
3. What is the largest k for which G has a clique of size k?
4. What is the largest k for which G has an independent set of size k?
5. What is the shortest path from vertex 7 to vertex 6

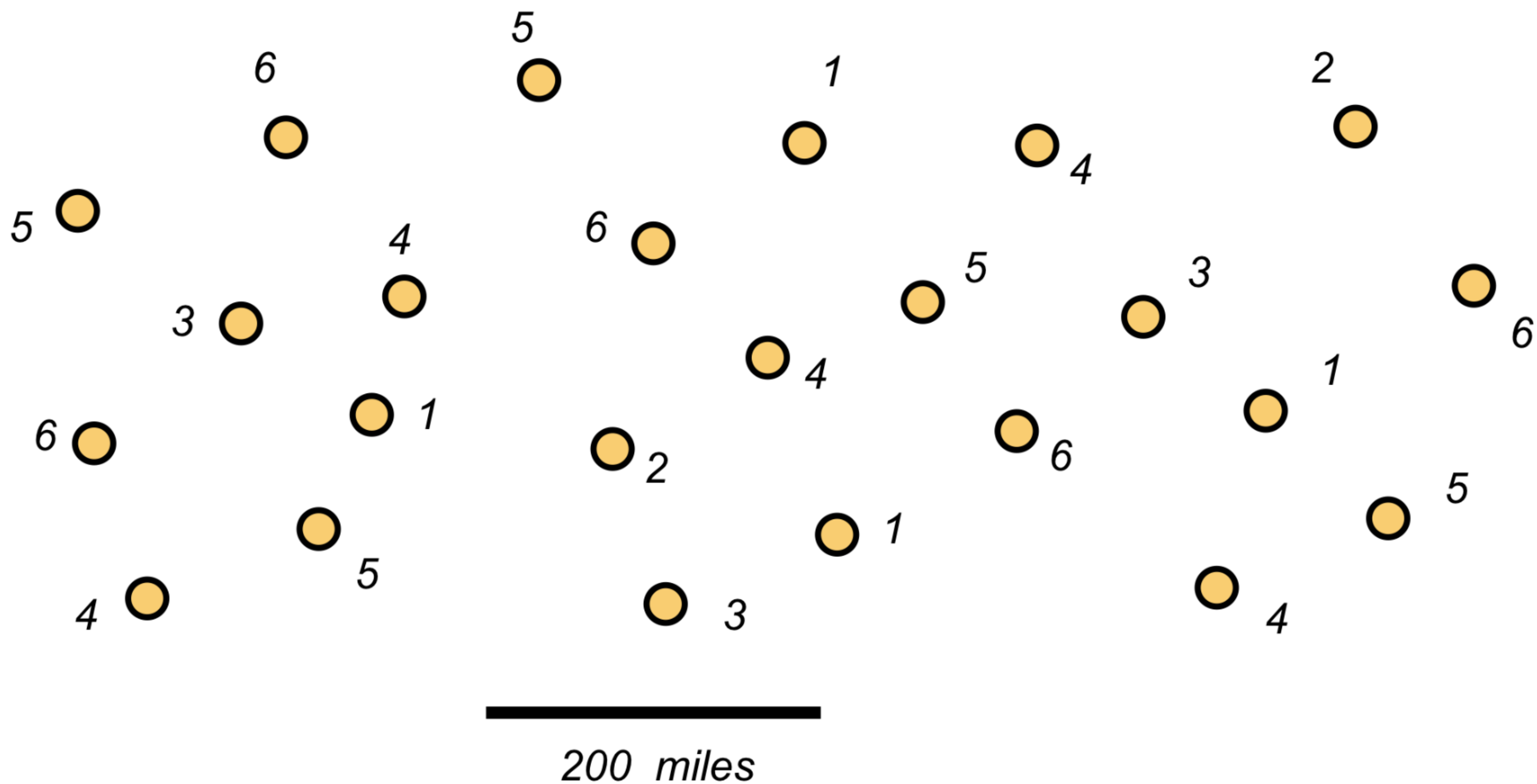



**Figure 1.4.:** Radio Stations

Example 1.2. In Figure 1.4, we show the location of some radio stations in the plane, together with a scale indicating a distance of 200 miles. Radio stations that are closer than 200 miles apart must broadcast on different frequencies to avoid interference

We've shown that 6 different frequencies are enough. Can you do better?

Can you find 4 stations each of which is within 200 miles of the other 3? Can you find 8 stations each of is more than 200 miles away from the other 7? Is there a natural way to define a graph associated with this problem?

Example 1.3. How big must an applied combinatorics class be so that there are either

(a) six students with each pair having taken at least one other class together, or (b) six students with each pair together in a class for the first time.Is this really a hard problem or can we figure it out in just a few minutes, scribbling on a napkin

# Combinatorics and Number Theory

Broadly, number theory concerns itself with the properties of the positive integers.

- G.H. Hardy was a brilliant British mathematician who lived through both World Wars and conducted a large deal of number-theoretic research. He was also a pacifist who was happy that, from his perspective, his research was not "useful". He wrote in his 1940 essay A Mathematician's Apology "

- “[n]o one has yet discovered any warlike purpose to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years.”

- Little did he know, the purest mathematical ideas of number theory would soon become indispensable for the cryptographic techniques that kept communications secure. Our subject here is not number theory, but we will see a few times where combinatorial techniques are of use in number theory

Example 1.4. Form a sequence of positive integers using the following rules. Start with a positive integer $n>1$. If $n$ is odd, then the next number is $3n+1$. If $n$ is even, then the next number is $n/2$. Halt if you ever reach 1. For example, if we start with 28, the sequence is

For example, if we start with 28, the sequence is
28,14,7,22,11,34,17,52,26,13,40,20,10,5,16,8,4,2,1.

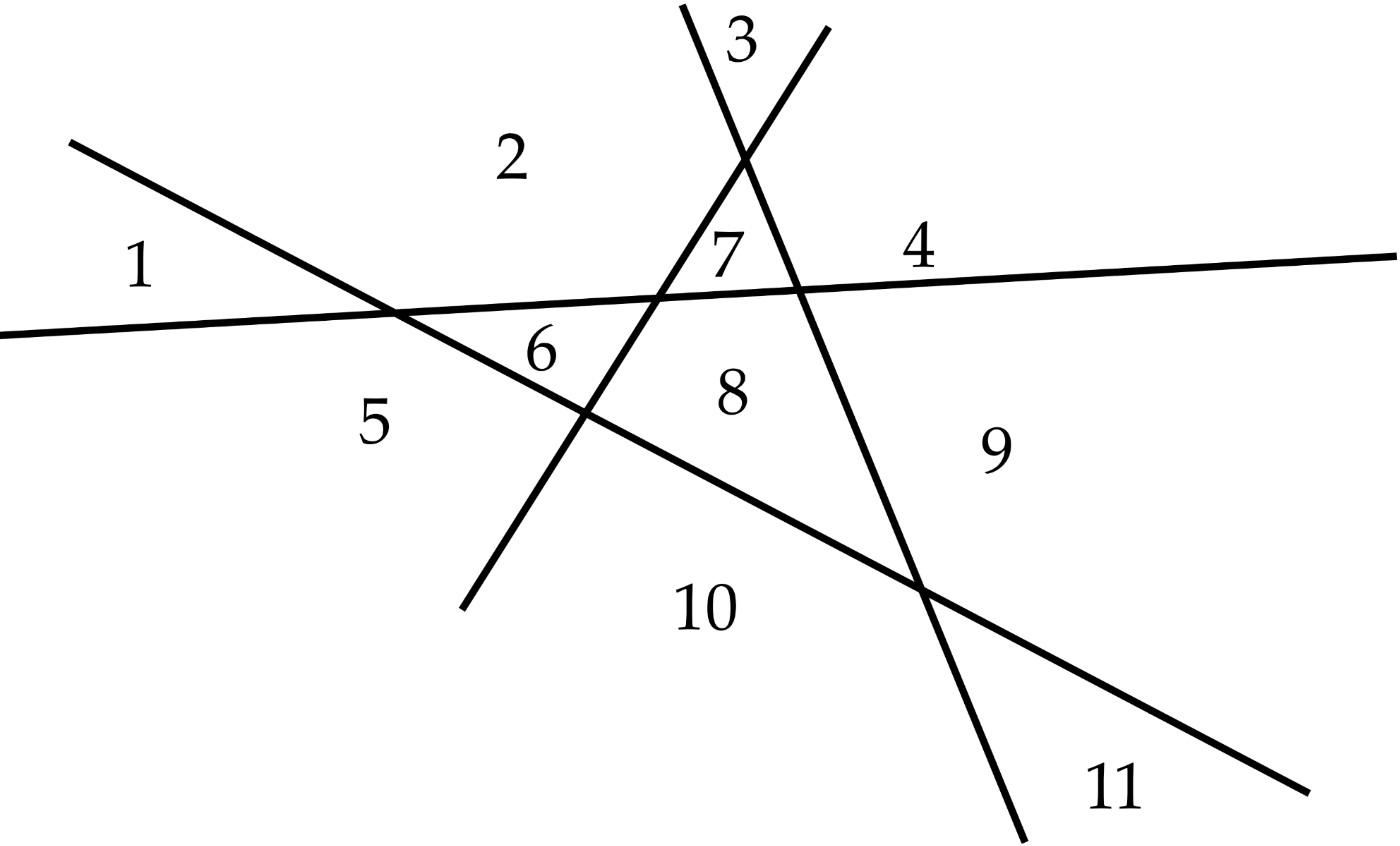
Now suppose you start with 19. Then the first few terms are
19,58,29,88,44,22

But now we note that the integer 22 appears in the first sequence, so the two sequences will agree from this point on. Sequences formed by this rule are called Collatz sequences

Pick a number somewhere between 100 and 200 and write down the sequence you get. Regardless of your choice, you will eventually halt with a 1. However, is there some positive integer n (possibly quite large) so that if you start from n, you will never reach 1

# Combinatorics and Geometry

Example 1.7. In Figure 1.5, we show a family of 4 lines in the plane. Each pair of lines intersects and no point in the plane belongs to more than two lines. These lines determine 11 regions


3
2
1
7
4
6
8
5
9
10
11

Under these same restrictions, how many regions would a family of 8947 lines de-termine? Can different arrangements of lines determine different numbers of regions?

# Combinatorics and Optimization

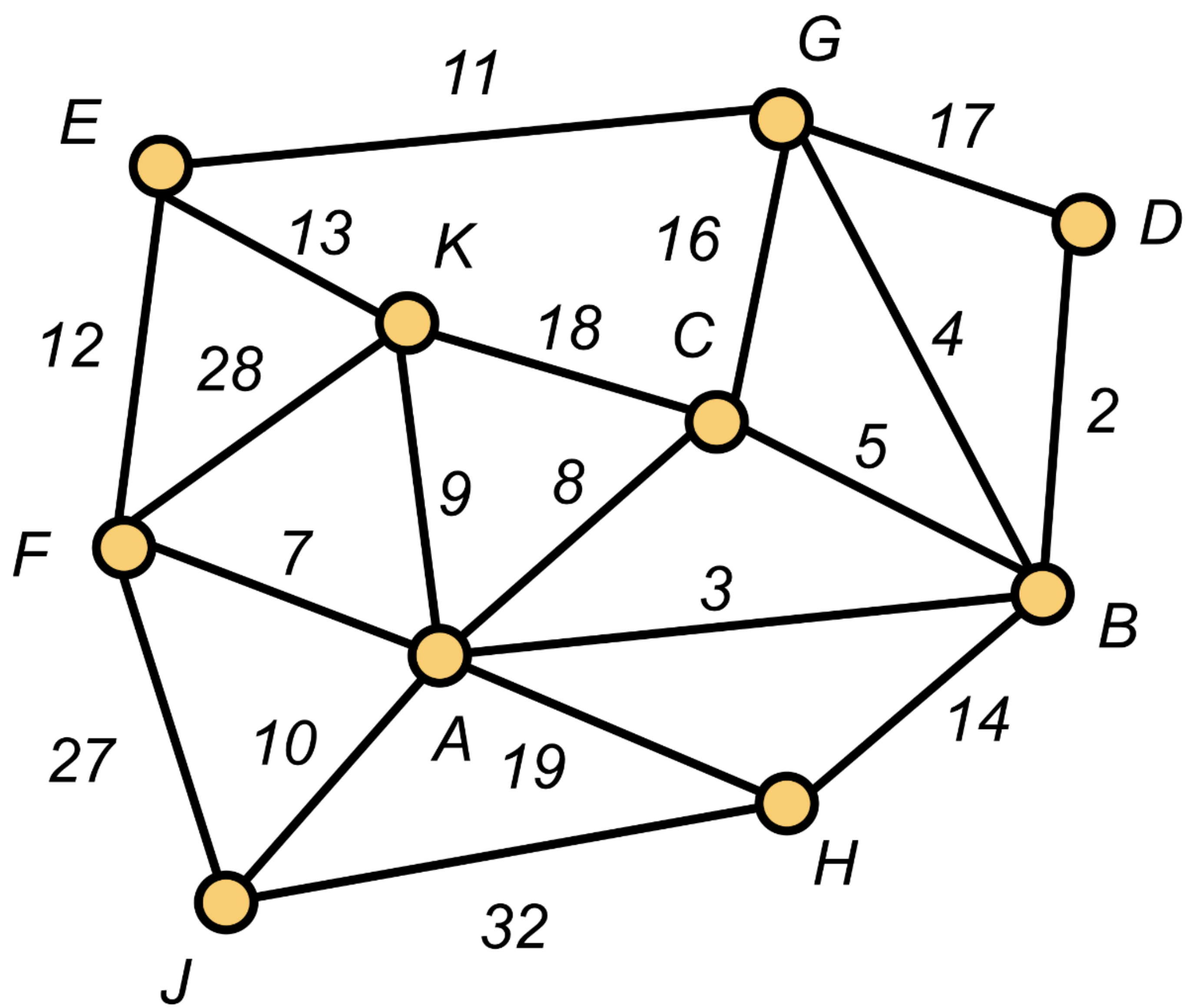



**Figure 1.7.:** A LABELED GRAPH WITH WEIGHTED EDGES

- Q1:What is the shortest path from vertex E to vertex B?
Suppose Ariel is a salesperson whose home base is city A.
Q2:In what order should Ariel visit the other cities so that she goes through each of them at least once and returns home at the end—while keeping the total distance

traveled to a minimum? Can Ariel accomplish such a tour visiting each city exactly once?

Sanjay is a highway inspection engineer and must traverse every highway each month. Sanjay's homebase is City E.

Q3:In what order should Sanjay traverse the highways to minimize the total distance traveled? Can Sanjay make such a tour traveling along each highway exactly once?

Example 1.11. Now suppose that the vertices are locations of branch banks in Atlanta and that the weights on an edge represents the cost, in millions of dollars, of building a high capacity data link between the branch banks at it two end points. In this model, if there is no edge between two branch banks, it means that the cost of building a data link between this particular pair is prohibitively high (here we might be tempted to say the cost is infinite, but the authors don't admit to knowing the meaning of this word)

Our challenge is to decide which data links should be constructed to form a network in which any branch bank can communicate with any other branch. We assume that data can flow in either direction on a link, should it be built, and that data can be relayed through any number of data links. So to allow full communication, we should construct a spanning tree in this network. In Figure 1.8, we show a graph G on the left and one of its many spanning trees on the right

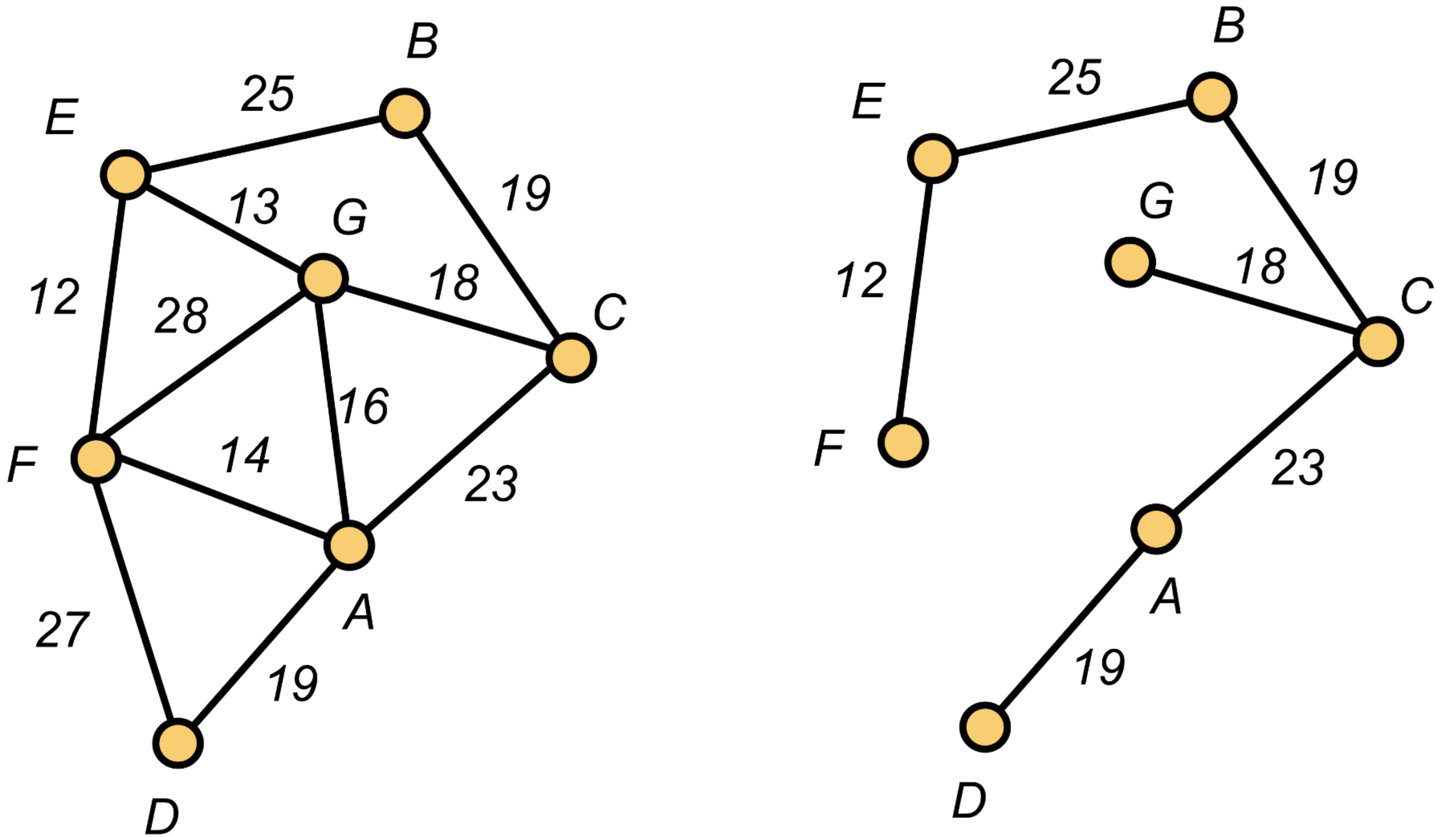



**Figure 1.8.:** A WEIGHTED GRAPH AND SPANNING TREE

Of all spanning trees, the bank would naturally like to find one having minimum weight

# Sudoku Puzzles

Example 1.12. A Sudoku puzzle is a 9×9 array of cells that when completed have the
integers 1,2,...,9 appearing exactly once in each row and each column. Also (and this
is what makes the puzzles so fascinating), the numbers 1, 2, 3,...,9 appear once in
each of the nine 3×3 subquares identified by the darkened borders. To be considered
a legitimate Sudoku puzzle, there should be a unique solution. In Figure 1.9, we show
two Sudoku puzzles. The one on the right is fairly easy, and the one on the left is far
more challenging

There are many sources of Sudoku puzzles, and software that generates Sudoku puzzles and then allows you to play them with an attractive GUI is available for all operating systems we know anything about (although not recommend to play them during class!). Also, you can find Sudoku puzzles on the web at On this site, the “Evil” ones are just that

How does Rory make up good Sudoku puzzles, ones that are difficult for Mandy to
solve? How could Mandy use a computer to solve puzzles that Rory has constructed?
What makes some Sudoku puzzles easy and some of them hard?
The size of a Sudoku puzzle can be expanded in an obvious way, and many newspa-
pers include a $16 \times 16$ Sudoku puzzle in their Sunday edition (just next to a challenging
crosswords puzzle). How difficult would it be to solve a $1024 \times 1024$ Sudoku puzzle,
even if you had access to a powerful computer

- Sudoku associative memory