Mathematical logic

Chapter 2

NOTES ON DISCRETE MATHEMATICS

2.1 The basic picture

Reality		Model		Theory
herds of sheep				
piles of rocks	$\rightarrow$	$\mathbb{N} = \{0, 1, 2, \ldots\}$	$\rightarrow$	$\forall x : \exists y : y = x + 1$
tally marks				

Axioms, models, and inference rules

- A list of **axioms** that are true statements about the model
- A list of **inference rules**
- **Derive new true statements from the axioms**

THEORY

- The axioms and inference rules together generate a **theory**
- **A Theory** that consists of all statements that can be constructed from the axioms by applying the inference rules.

Example

- All fish are green (axiom). George Washington is a fish (axiom).
- From “all X are Y” and “Z is X”, we can derive “Z is Y” (inference rule). Thus George Washington is green (theorem).
- Theories are attempts to describe **models**. A model is typically a collection of objects and relations between them.

Consistency

- A theory is **consistent** if it can't prove both P and not-P for any P.
- Too many axioms,
  - you can get an inconsistency: "All fish are green; all sharks are not green; all sharks are fish; George Washington is a shark" gets us into trouble pretty fast.
- If we don't throw in enough axioms, we under-constrain the model.

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The Peano axioms for the natural numbers

- There is a number 0 and that any number x has a successor S(x) (think of S(x) as x + 1).

- If we stop there, we might have a model that contains only 0, with S(0) = 0. If we add in $0 \neq S(x)$ for any x, then we can get stuck at S(0) = 1 = S(1).

- If we add yet another axiom that says S(x) = S(y) if and only if x = y, then we get all the ordinary natural numbers 0,S(0) = 1,S(1) = 2, etc., but we could also get some extras: say $0'$, $S(0') = 1'$, $S(1') = 0'$.

- 0 is a natural number.
- The next four axioms describe the equality relation. Since they are logically valid in first-order logic with equality, they are not considered to be part of "the Peano axioms" in modern treatments.[5]
- For every natural number x, $x = x$. That is, equality is reflexive.
- For all natural numbers x and y, if $x = y$, then $y = x$. That is, equality is symmetric.
- For all natural numbers x, y and z, if $x = y$ and $y = z$, then $x = z$. That is, equality is transitive.
- For all a and b, if b is a natural number and $a = b$, then a is also a natural number. That is, the natural numbers are closed under equality.
- The remaining axioms define the arithmetical properties of the natural numbers. The naturals are assumed to be closed under a single-valued "successor" function S.
- For every natural number n, $S(n)$ is a natural number.
- For all natural numbers m and n, $m = n$ if and only if $S(m) = S(n)$. That is, S is an injection.
- For every natural number n, $S(n) = 0$ is false. That is, there is no natural number whose successor is 0.

The language of logic

- The basis of mathematical logic is **propositional logic**
- The model is a collection of **statements** that are either true or false.
  - An axiom : George Washington is a fish
  - To prove the truth of more complicated statements
    - George Washington is a fish or 2+2=5

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predicate logic

- **constants** (stand-ins for objects in the model like “George Washington”) and **predicates** (stand-ins for properties like “is a fish”)
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Statements

- For all x, if x is a fish then x is green.
- As a bonus, we usually get functions ("f(x) = the number of books George Washington owns about x")
- Equality ("George Washington = 12" implies "George Washington + 5 = 17")
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Standard axiom systems and models

- The natural numbers N.
- The integers Z.
- The rational numbers Q.
- The real numbers R.
- The complex numbers C.
- The universe of sets.
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2.2 Propositional logic

- **Propositional logic** is the simplest form of logic
- the only statements that are considered are **propositions**, which contain no variables.
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- Examples of propositions:
  - 2 + 2 = 4. (Always true).
  - 2 + 2 = 5. (Always false).

- Examples of non-propositions:

- - x+2=4. (May be true, may not be true; it depends on the value of x.)
  - $x \cdot 0 = 0$. (Always true, but it’s still not a proposition because of the variable.)

- $x \cdot 0 = 1$. (Always false, but not a proposition because of the variable.)

Operations on propositions

- **Negation**
    - The **negation** of p is written as ¬p, or sometimes ~p, −p or p. It has the property that it is false when p is true, and true when p is false.
- **Or**
    - The **or** of two propositions p and q is written as p ∨ q, and is true as long as at least one, or possibly both, of p and q is true.
- **Exclusive or**
    - If you want to exclude the possibility that both p and q are true, you can use **exclusive or** instead. This is written as p ⊕ q, and is true precisely when exactly one of p or q is true.
- **And**
    - The **and** of p and q is written as p∧q, and is true only when both p and q are true.[3]
    - “(2 + 2 = 4) ∧ (3 + 3 = 6).”
- **Implication**
    - This is the most important connective for proofs. An **implication** represents an “if. . . then” claim. If p implies q, then we write p → q or p ⇒ q, depending on our typographic convention and the availability of arrow symbols in our favorite font.

- In fact, the only way for $p \to q$ to be *false* is for p to be true but q to be false. Because of this, $p \to q$ can be rewritten as $\neg p \vee q$.

- So, for example, the statements "If 2 + 2 = 5, then I'm the Pope",

- "If I'm the Pope, then 2 + 2 = 4", and

- "If 2 + 2 = 4, then 3 + 3 = 6", are all true, provided the if/then is interpreted as implication.

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NOT p	$\neg p$	$\bar{p}, \sim p$
p AND q	$p \wedge q$	
p XOR q	$p \oplus q$	
p OR q	$p \vee q$	
p implies q	$p \rightarrow q$	$p \Rightarrow q$, $p \supset q$
p if and only if q	$p \leftrightarrow q$	$p \Leftrightarrow q$

- Table 2.1: Compound propositions. The rightmost column gives alternate forms. Precedence goes from strongest for ¬ to weakest for ↔

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Biconditional

- **Biconditional**
  - Suppose that $p \rightarrow q$ and $q \rightarrow p$, so that either both p and q are true or both p and q are false. In this case, we write $p \leftrightarrow q$ or $p \Leftrightarrow q$, and say that p holds **if and only if** q holds.
- The truth of $p \leftrightarrow q$ is still just a function of the truth or false hood of p and q; though there doesn't need to be any connection between the two sides of the statement, "2+2 = 5 if and only if I am the Pope" is a true statement (provided it is not uttered by the Pope). The only way for $p \leftrightarrow q$ to be false is for one side to be true and one side to be false.

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compound proposition

- The result of applying any of these operations is called a **compound proposition**.
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Precedence

- precedence in C-like programming languages
- Examples: $(\neg p \vee q \wedge r \rightarrow s \leftrightarrow t)$ is interpreted as $((((\neg p) \vee (q \wedge r)) \rightarrow s) \leftrightarrow t)$
- Both OR and AND are associative, so $(p \vee q \vee r)$ is the same as $((p \vee q) \vee r)$ and as $(p \vee (q \vee r))$, and similarly $(p \wedge q \wedge r)$ is the same as $((p \wedge q) \wedge r)$ and as $(p \wedge (q \wedge r))$
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- Note that this convention is not universal: many mathematicians give AND and OR equal precedence, so that the meaning of $p \land q \lor r$ is ambiguous without parentheses.

- $a \to b \to c$ is read as $a \to (b \to c)$. Except for type theorists and Haskell programmers, few people ever remember this, so it is usually safest to put in the parentheses.

Truth tables

p	$\neg p$
0	1
1	0

- We can think of each row of a truth table as a model for propositional logic, since the only things we can describe in propositional logic are whether particular propositions are true or not. Constructing a truth table corresponds to generating all possible models.
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p	q	$p \vee q$	$p \oplus q$	$p \wedge q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	1	1	0	1	0
1	0	1	1	0	0	0
1	1	1	0	1	1	1

- Proving a proposition using a truth table is a simple version of **model checking**: we enumerate all possible **models** of a given collection of simple propositions, and see if what we want to prove holds in all models

- For predicate logic, model checking becomes more complicated, because a typical system of axioms is likely to have infinitely many models, many of which are likely to be infinitely large. There we will need to rely much more on proofs constructed by applying inference rules.

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Tautologies and logical equivalence

- A compound proposition that is true no matter what the truth-values of the propositions it contains is called a **tautology**.

- For example, $p \rightarrow p$, $p \vee \neg p$, and $\neg(p \wedge \neg p)$ are all tautologies, as can be verified by constructing truth tables.

- If a compound proposition is always false, it's a **contradiction**. The negation of a tautology is a contradiction and vice versa

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- The most useful class of tautologies are **logical equivalences**.

- This is a tautology of the form $X \leftrightarrow Y$, where X and Y are compound propositions. In this case, X and Y are said to be **logically equivalent** and we can substitute one for the other in more complex propositions.

- We write $X \equiv Y$ if X and Y are logically equivalent.

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- Boolean formulas that equality does for algebraic formulas: if we know (for example), that $p \vee \neg p$ is equivalent to 1, and $q \vee 1$ is equivalent to 1, we can grind $q \vee p \vee \neg p \equiv q \vee 1 \equiv 1$ without having to do anything particularly clever.
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- $p \vee p \equiv p$: Use the truth table

p	$p \vee p$
0	0
1	1

- $p \to q \equiv \neg p \vee q$: Again construct a truth table

p	q	$p \to q$	$\neg p \vee q$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

- $\neg(p \vee q) \equiv \neg p \wedge \neg q$: (one of De Morgan's laws; the other is $\neg(p \wedge q) \equiv \neg p \vee \neg q$).

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
0	0	0	**1**	1	1	**1**
0	1	1	**0**	1	0	**0**
1	0	1	**0**	0	1	**0**
1	1	1	**0**	0	0	**0**

- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (one of the distributive laws; the other is $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$).

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	**0**	0	0	**0**
0	0	1	0	**0**	0	1	**0**
0	1	0	0	**0**	1	0	**0**
0	1	1	1	**1**	1	1	**1**
1	0	0	0	**1**	1	1	**1**
1	0	1	0	**1**	1	1	**1**
1	1	0	0	**1**	1	1	**1**
1	1	1	1	**1**	1	1	**1**

- $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$. Now things are getting messy, so building a full truth table may take a while. But we have take a shortcut by using logical equivalences that we've already proved (plus associativity of $\vee$):

$$
\begin{aligned}
(p \to r) \vee (q \to r) &\equiv (\neg p \vee r) \vee (\neg q \vee r) && \text{[Using } p \to q \equiv \neg p \vee q \text{ twice]} \\
&\equiv \neg p \vee \neg q \vee r \vee r && \text{[Associativity and commutativity of } \vee\text{]} \\
&\equiv \neg p \vee \neg q \vee r && [p \equiv p \vee p] \\
&\equiv \neg(p \wedge q) \vee r && \text{[De Morgan's law]} \\
&\equiv (p \wedge q) \to r. && [p \to q \equiv \neg p \vee q]
\end{aligned}
$$

$\neg\neg p \equiv p$	Double negation
$\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's law
$\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's law
$p \wedge q \equiv q \wedge p$	Commutativity of AND
$p \vee q \equiv q \vee p$	Commutativity of OR
$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	Associativity of AND
$p \vee (q \vee r) \equiv (p \vee q) \vee r$	Associativity of OR
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	AND distributes over OR
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	OR distributes over AND
$p \rightarrow q \equiv \neg p \vee q$	Equivalence of implication and OR
$p \rightarrow q \equiv \neg q \rightarrow \neg p$	Contraposition
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	Expansion of if and only if
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$	Inverse of if and only f
$p \leftrightarrow q \equiv q \leftrightarrow p$	Commutativity of if and only if

Table 2.2: Common logical equivalences (see also [Fer08, Theorem 1.1])

- **contrapositive** of $p \rightarrow q$ is $\neg q \rightarrow \neg p$

- it is logically equivalent to the original implication. For example, the contrapositive of “If I am human then I am a mammal” is “If I am not a mammal then I am not human”.

- A **proof by contraposition** demonstrates that p implies q by assuming $\neg q$ and then proving $\neg p$; it is similar but not identical to an **indirect proof**, which assumes $\neg p$ and derives a contradiction.

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- The **inverse** of $p \to q$ is $\neg p \to \neg q$

- There is often no connection between the truth of an implication and the truth of its inverse: “If I am human then I am a mammal” does not have the same truth-value as “If I am not human then I am not a mammal,”

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- The **converse** of $p \rightarrow q$ is $q \rightarrow p$.

- the converse of “If I am human then I am a mammal” is “If I am a mammal then I am human.” The converse of a statement is always logically equivalent to the inverse.

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Equivalences involving true and false

- law of the excluded middle
  - $P \vee \neg P \equiv 1$
- law of non-contradiction
  - $P \wedge \neg P \equiv 0$.
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absorption laws

$$P \wedge 0 \equiv 0 \qquad P \vee 0 \equiv P$$

$$P \wedge 1 \equiv P \qquad P \vee 1 \equiv 1$$

$$P \leftrightarrow 0 \equiv \neg P \qquad P \oplus 0 \equiv P$$

$$P \leftrightarrow 1 \equiv P \qquad P \oplus 1 \equiv \neg P$$

$$P \rightarrow 0 \equiv \neg P \qquad 0 \rightarrow P \equiv 1$$

$$P \rightarrow 1 \equiv 1 \qquad 1 \rightarrow P \equiv P$$

Table 2.3: Absorption laws. The first four are the most important. Note that $\wedge$, $\vee$, $\oplus$, and $\leftrightarrow$ are all commutative, so reversed variants also work.

$$
\begin{aligned}
(P \wedge (P \to Q)) \to Q &\equiv (P \wedge (\neg P \vee Q)) \to Q && \text{expand } \to \\
&\equiv ((P \wedge \neg P) \vee (P \wedge Q)) \to Q && \text{distribute } \vee \text{ over } \wedge \\
&\equiv (0 \vee (P \wedge Q)) \to Q && \text{non-contradiction} \\
&\equiv (P \wedge Q) \to Q && \text{absorption} \\
&\equiv \neg(P \wedge Q) \vee Q && \text{expand } \to \\
&\equiv (\neg P \vee \neg Q) \vee Q && \text{De Morgan's law} \\
&\equiv \neg P \vee (\neg Q \vee Q) && \text{associativity} \\
&\equiv \neg P \vee 1 && \text{excluded middle} \\
&\equiv 1 && \text{absorption}
\end{aligned}
$$

Normal forms

- A compound proposition is in **conjunctive normal form** (**CNF** for short) if it is obtained by ANDing together ORs of one or more variables or their negations

- P, $(P \vee Q) \wedge R$, $(P \vee Q) \wedge (Q \vee R) \wedge (\neg P)$, and $(P \vee Q) \wedge (P \vee \neg R) \wedge (\neg P \vee Q \vee S \vee T \vee \neg U)$ are in CNF

- $(P \vee (Q \wedge R)) \wedge (P \vee \neg R) \wedge (\neg P \vee Q)$ are not

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- A famous Zen koan involves a student going for instruction to a sword master who also happens to be a Zen monk. The master tells the student “If you draw your sword, I will cut off your head. If you do not draw your sword, I will cut off your head.” How should the student interpret this alarming statement?

- Writing P for the proposition that the student draws his sword and Q for the proposition that the master cuts off his head, we can immediately convert this to CNF by expanding the implications:

- $(P \rightarrow Q) \wedge (\neg P \rightarrow Q) \equiv (\neg P \vee Q) \wedge (P \vee Q)$

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- $(\neg P \vee Q) \wedge (P \vee Q) \equiv (\neg P \wedge P) \vee (\neg P \wedge Q) \vee (Q \wedge P) \vee (Q \wedge Q) \equiv 0 \vee (\neg P \wedge Q) \vee (Q \wedge P) \vee Q \equiv (\neg P \wedge Q) \vee (Q \wedge P) \vee Q$.

- Now the proposition is in **disjunctive normal form**, which means it's an OR of ANDs.

- $(P \vee Q) \wedge (\neg P \vee R) \to Q \vee R$

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- Similarly, a compound proposition is in **disjunctive normal form** (**DNF**) if it consists of an OR of ANDs, e.g. $(P \wedge Q) \vee (P \wedge \neg R) \vee (\neg P \wedge Q)$

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2.3 Predicate（述詞) logic

- Socrates is a man.
- If Socrates is a man, then Socrates is mortal.
- Therefore, Socrates is mortal.
- This is an application of the inference rule called **modus ponens**, which says that from p and p → q you can deduce q. The first two statements are axioms (meaning we are given them as true without proof), and the last is the conclusion of the argument.

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Variables and predicates

- • "x is human."
  • "x is the parent of y." • "$x+2=x^2$."

- These are not propositions because they have variables in them. Instead, they are **predicates**; statements whose truth-value depends on what concrete object takes the place of the variable.

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- Predicates are often abbreviated by single capital letters followed by a list of **arguments**, the variables that appear in the predicate, e.g.:
  - H(x) = “x is human.”
  - P(x,y) = “x is the parent of y.” •
  - Q(x)=“$x+2=x^2$.”
- H(Spocrates) = “Spocrates is human.”
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- In **first-order logic**, which is what we will be using in this course, variables always refer to things and never to predicates: any predicate symbol is effectively a constant. There are higher-order logics that allow variables to refer to predicates, but most mathematics accomplishes the same thing by representing predicates with sets

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Quantifiers

- **bind** the variables using **quantifiers**, which state whether the claim we are making applies to all values of the variable (**universal quantification**), or whether it may only apply to some (**existential quantification**)
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Universal quantifier

- The **universal quantifier** ∀ (pronounced "for all") says that a statement must be true for all values of a variable within some *universe* of allowed values (which is often implicit).

- For example,

  - "all humans are mortal" could be written $\forall x : \mathrm{Human}(x) \rightarrow \mathrm{Mortal}(x)$

  - "if x is positive then x + 1 is positive" could be written $\forall x : x > 0 \rightarrow x + 1 > 0$.

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- If you want to make the universe explicit, use set membership notation.
  - An example would be
    - $\forall x \in Z : x > 0 \rightarrow x+1 > 0$.
  - This is logically equivalent to writing
    - $\forall x : x \in Z \rightarrow (x > 0 \rightarrow x + 1 > 0)$
    - or to writing $\forall x : (x \in Z \wedge x > 0) \rightarrow x + 1 > 0$, but the short form makes it more clear that the intent of $x \in Z$ is to restrict the range of x.

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- The statement $\forall x : P(x)$ is equivalent to a very large AND; for example, $\forall x \in N : P(x)$ could be rewritten (if you had an infinite amount of paper) as $P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge \ldots$.

- Normal first-order logic doesn't allow infinite expressions like this, but it may help in visualizing what $\forall x : P(x)$ actually means.

- Another way of thinking about it is to imagine that x is supplied by some adversary and you are responsible for showing that P(x) is true; in this sense, the universal quantifier chooses the *worst case* value of x.

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Existential quantifier

- The **existential quantifier** ∃ (pronounced “there exists”) says that a statement must be true for at least one value of the variable. So “some human is mortal” becomes $\exists x$: Human(x) $\wedge$ Mortal(x).
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- Note that we use AND rather than implication here; the statement $\exists x$: Human(x) → Mortal(x) makes the much weaker claim that "there is some thing x, such that if x is human, then x is mortal," which is true in any universe that contains an immortal purple penguin—since it isn't human, Human(penguin) → Mortal(penguin) is true.
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- As with $\forall$, $\exists$ can be limited to an explicit universe with set membership notation, e.g., $\exists x \in Z : x = x^2$. This is equivalent to writing $\exists x : x \in Z \wedge x = x^2$.

- The formula $\exists x : P(x)$ is equivalent to a very large OR, so that $\exists x \in N : P(x)$ could be rewritten as $P(0) \vee P(1) \vee P(2) \vee P(3) \vee \ldots$. Again, you can't generally write an expression like this if there are infinitely many terms, but it gets the idea across.

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Negation and quantifiers

- $\neg\forall x : P(x) \equiv \exists x : \neg P(x)$.
- $\neg\exists x : P(x) \equiv \forall x : \neg P(x)$.
- These are essentially the quantifier version of De Morgan's laws: the first says that if you want to show that not all humans are mortal, it's equivalent to finding some human that is not mortal. The second says that to show that no human is mortal, you have to show that all humans are not mortal.
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Restricting the scope of a quantifier

$$\forall x : x > 0 \to x - 1 \geq 0$$

or in abbreviated form by including the restriction in the quantifier expression itself:

$$\forall x > 0 : x - 1 \geq 0.$$

Similarly

$$\exists x : x > 0 \wedge x^2 = 81$$

can be written as

$$\exists x > 0 : x^2 = 81.$$

$$\neg \exists x \in \mathbb{Z} : x^2 = 79$$

which is interpreted as

$$\neg \exists x : (x \in \mathbb{Z} \wedge x^2 = 79)$$

or, equivalently

$$\forall x : x \in \mathbb{Z} \rightarrow x^2 \neq 79.$$

Nested quantifiers

- It is possible to nest quantifiers, meaning that the statement bound by a quantifier itself contains quantifiers. For example, the statement “there is no largest prime number” could be written as

- $\neg\exists x : (\mathrm{Prime}(x) \wedge \forall y : y > x \rightarrow \neg\mathrm{Prime}(y))$
  i.e., “there does not exist an x that is prime and any y greater than x is not prime.” Or in a shorter (though not strictly equivalent) form: $\forall x \exists y : y > x \wedge \mathrm{Prime}(y)$

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- If we write likes(x, y) for the predicate that x likes y, the statements
    - $\forall x \exists y$: likes(x, y)
    - $\exists y \forall x$: likes(x, y)

- The first says that for every person, there is somebody that that person likes: we live in a world with no complete misanthropes.

- The second says that there is some single person who is so immensely popular that everybody in the world likes them.

- The nesting of the quantifiers is what makes the difference: in $\forall x \exists y$: likes(x, y), we are saying that no matter who we pick for x, $\exists y$: likes(x, y) is a true statement; while in $\exists y \forall x$: likes(x, y), we are saying that there is some y that makes $\forall x$: likes(x, y) true.

- for example $\forall x \forall y : (x = y \rightarrow x+1 = y+1)$ is logically equivalent to $\forall y \forall x : (x = y \rightarrow y + 1 = x + 1)$,

- but $\forall x \exists y : y < x$ is not logically equivalent to $\exists y \forall x : y < x$.

$$\left[\lim_{x\to\infty} f(x) = y\right] \equiv \left[\forall \epsilon > 0 : \exists N : \forall x > N : |f(x) - y| < \epsilon\right].$$

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1. The adversary picks ϵ, which must be greater than 0.
2. We pick N.
3. The adversary picks x, which must be greater than N.
4. We win if $f(x)$ is within ϵ of y.

So, for example, a proof of

$$\lim_{x\to\infty} 1/x = 0$$

$$\left[\lim_{x\to\infty} f(x) = y\right] \equiv \left[\forall\epsilon > 0 : \exists N : \forall x > N : |f(x) - y| < \epsilon\right].$$

$$\lim_{x\to\infty} 1/x = 0$$

- would follow exactly this game plan:

1. Choose some $\epsilon > 0$.
2. Let $N > 1/\epsilon$. (Note that we can make our choice depend on previou choices.)
3. Choose any $x > N$.
4. Then $x > N > 1/\epsilon > 0$, so $1/x < 1/N < \epsilon \to |1/x - 0| < \epsilon$. QED!

Here we give some more examples of translating English into statements in predicate logic.

All crows are black. $\forall x : \text{Crow}(x) \rightarrow \text{Black}(x)$

The formula is logically equivalent to either of

$$\neg\exists x \text{Crow}(x) \land \neg\text{Black}(x)$$

or

$$\forall x : \neg\text{Black}(x) \rightarrow \neg\text{Crow}(x).$$

- The latter is the core of a classic “paradox of induction” in philosophy

- If seeing a black crow makes me think it’s more likely that all crows are black, shouldn’t seeing a logically equivalent non-black non-crow (e.g., a banana yellow AMC Gremlin) also make me think all non-black objects are non-crows, i.e., that all crows are black? The paradox suggests that logical equivalence works best for true/false and not so well for probabilities.

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Some cows are brown. $\exists x : \text{Cow}(x) \wedge \text{Brown}(x)$

No cows are blue. $\neg\exists x : \text{Cow}(x) \wedge \text{Blue}(x)$

- Some other equivalent versions:

$$\forall x : \neg(\text{Cow}(x) \wedge \text{Blue}(x)$$
$$\forall x : (\neg\text{Cow}(x) \vee \neg\text{Blue}(x))$$
$$\forall x : \text{Cow}(x) \rightarrow \neg\text{Blue}(x)$$
$$\forall x : \text{Blue}(x) \rightarrow \neg\text{Cow}(x).$$

All that glitters is not gold. $\neg\forall x : \text{Glitters}(x) \rightarrow \text{Gold}(x)$

Or $\exists x : \text{Glitters}(x) \wedge \neg\text{Gold}(x)$. Note that the English syntax is a bit ambiguous: a literal translation might look like $\forall x : \text{Glitters}(x) \rightarrow \neg\text{Gold}(x)$, which is *not* logically equivalent. This is an example of how predicate logic is often more precise than natural language.

No shirt, no service.	$\forall x : \neg\text{Shir}t(x) \rightarrow \neg\text{Served}(x)$
Every event has a cause.	$\forall x \exists y : \text{Causes}(y, x)$

And a more complicated statement: Every even number greater than 2 can be expressed as the sum of two primes.

$$\forall x : (\text{Even}(x) \wedge x > 2) \rightarrow (\exists p \exists q : \text{Prime}(p) \wedge \text{Prime}(q) \wedge (x = p + q))$$

The last one is **Goldbach's conjecture**. The truth value of this statement is currently unknown.