Lecture 5: Graphs

- A **graph** is a structure in which pairs of **vertices** are connected by **edges.**

- Each edge may act like an ordered pair (in a **directed graph**) or an unordered pair (in an **undirected graph**). We’ve already seen directed graphs as a representation for relations. Most work in graph theory concentrates instead on undirected graphs.

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- In particular, unless otherwise specified, a *graph* will refer to a **finite simple undirected graph**:
  - an undirected graph with a finite number of vertices, where each edge connects two distinct vertices (thus no **self-loops**) and there is at most one edge between each pair of vertices (no **parallel edges**).
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Types of graphs

- Graphs are represented as ordered pairs G = (V,E), where V is a set of vertices and E a set of edges.

- The differences between different types of graphs depends on what can go in E. When not otherwise specified, we usually think of a *graph* as an *undirected graph* (see below), but there are other variants.

- Typically we assume that V and E are both finite.

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Directed graphs

- In a **directed graph** or *digraph*, each element of E is an ordered pair, and we think of edges as arrows from a **source**, **head**, or **initial vertex** to a **sink**, **tail**, or **terminal vertex**; each of these two vertices is called an **endpoint** of the edge.

- A directed graph is **simple** if there is at most one edge from one vertex to another.

- A directed graph that has multiple edges from some vertex u to some other vertex v is called a **directed multigraph**

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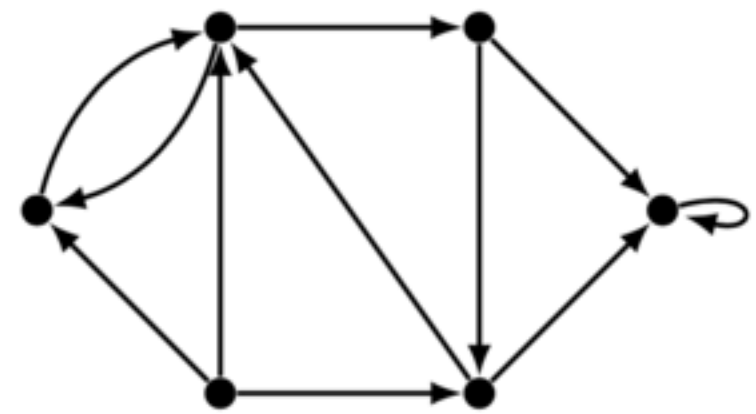

Figure 10.1: A directed graph

- For simple directed graphs, we can save a lot of ink by adopting the convention of writing an edge (u, v) from u to v as just uv.

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Undirected graphs

- In an **undirected graph**, each edge is an undirected pair, which we can represent as subset of V with one or two elements.

- A **simple undirected graph** contains no duplicate edges and no **loops** (an edge from some vertex u back to itself); this means we can represent all edges as two-element subsets of V .

- Most of the time, when we say *graph*, we mean a simple undirected graph. Though it is possible to consider infinite graphs, for convenience we will limit ourselves to finite graphs, where $n = |V|$ and $m = |E|$ are both natural numbers.

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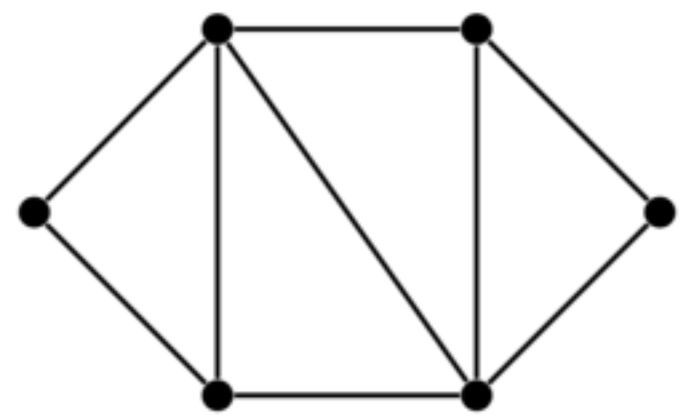
- 

  Figure 10.2: A graph

- As with directed graphs, instead of writing an edge as {u,v}, we will write an edge between u and v as just uv. Note that in an undirected graph, uv and vu are the same edge.

- If we have loops or parallel edges, we have a more complicated structure called a **multigraph**.

- This requires a different representation where elements of E are abstract edges and we have a function mapping each element of E to its endpoints.

- Some authors make a distinction between *pseudographs* (with loops) and *multigraphs* (without loops), but we'll use multigraph for both.

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- Simple undirected graphs also correspond to relations, with the restriction that the relation must be irreflexive (no loops) and symmetric (undirected edges).

- This also gives a representation of undirected graphs as directed graphs, where the edges of the directed graph always appear in pairs going in opposite directions.

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Hypergraphs

- In a **hypergraph**, the edges (called **hyperedges**) are arbitrary nonempty sets of vertices.
- A k-**hypergraph** is one in which all such hyperedges connected exactly k vertices; an ordinary graph is thus a 2-hypergraph.
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Hypergraphs

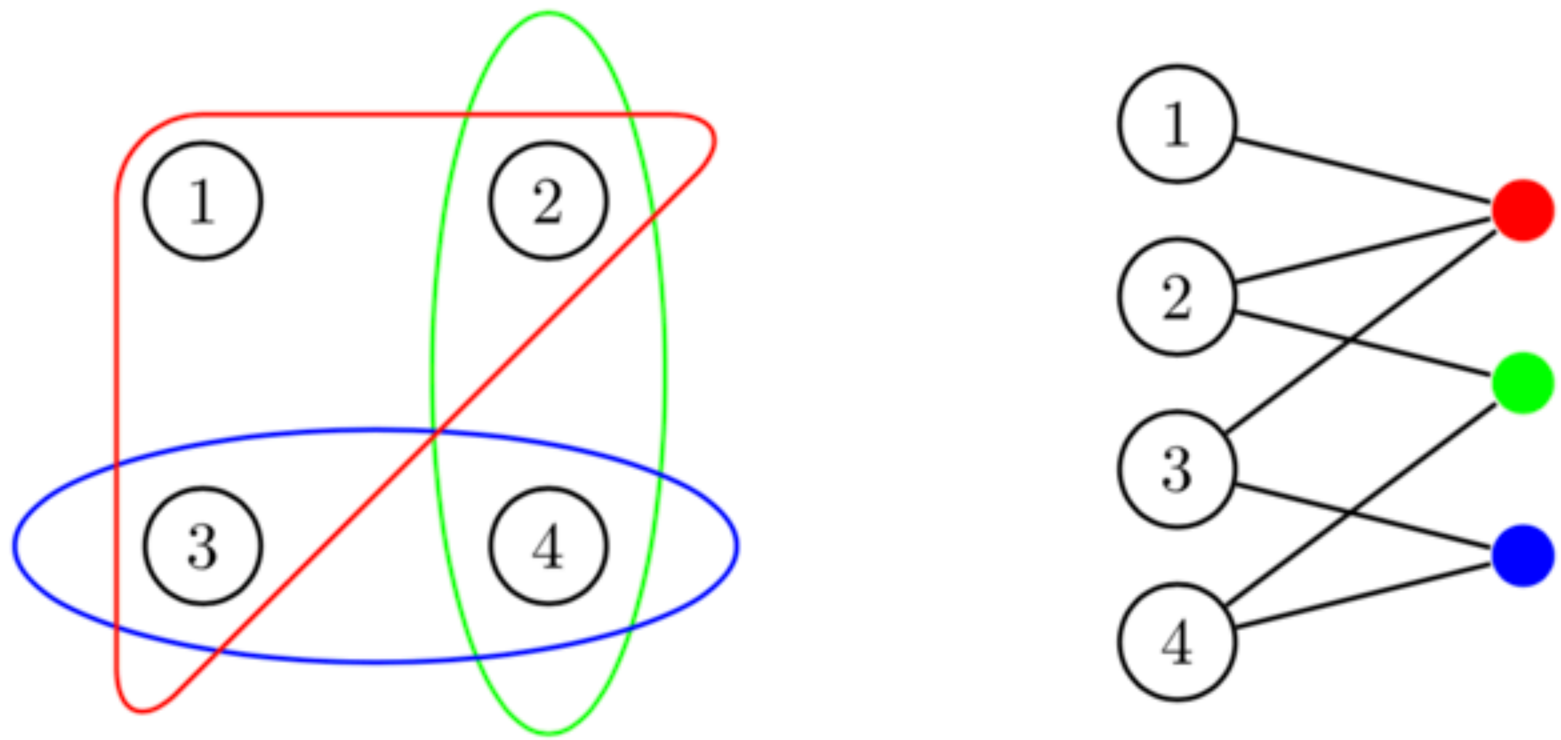

Figure 10.3: Two representations of a hypergraph. On the left, four vertices are connected by three hyperedges. On the right, the same four vertices are connected by ordinary edges to new vertices representing the hyperedges.

- Hypergraphs aren't used very much, because it is always possible (though not always convenient) to represent a hypergraph by a **bipartite graph**.

- In a bipartite graph, the vertex set can be partitioned into two subsets S and T , such that every edge connects a vertex in S with a vertex in T .

- To represent a hypergraph H as a bipartite graph, we simply represent the vertices of H as vertices in S and the hyperedges of H as vertices in T , and put in an edge (s,t) whenever s is a member of the hyperedge t in H. The right-hand side of Figure 10.3 gives an example.

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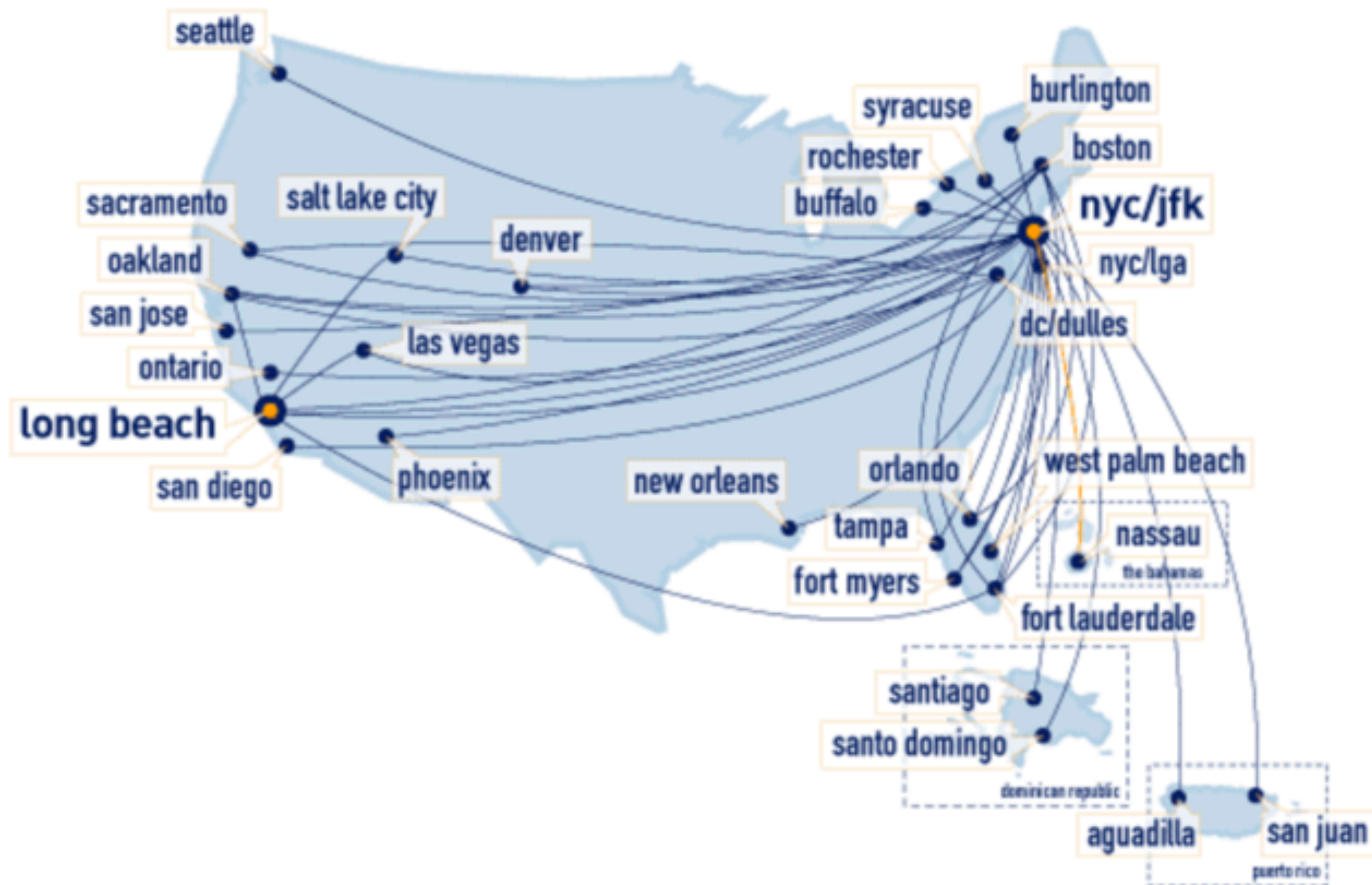
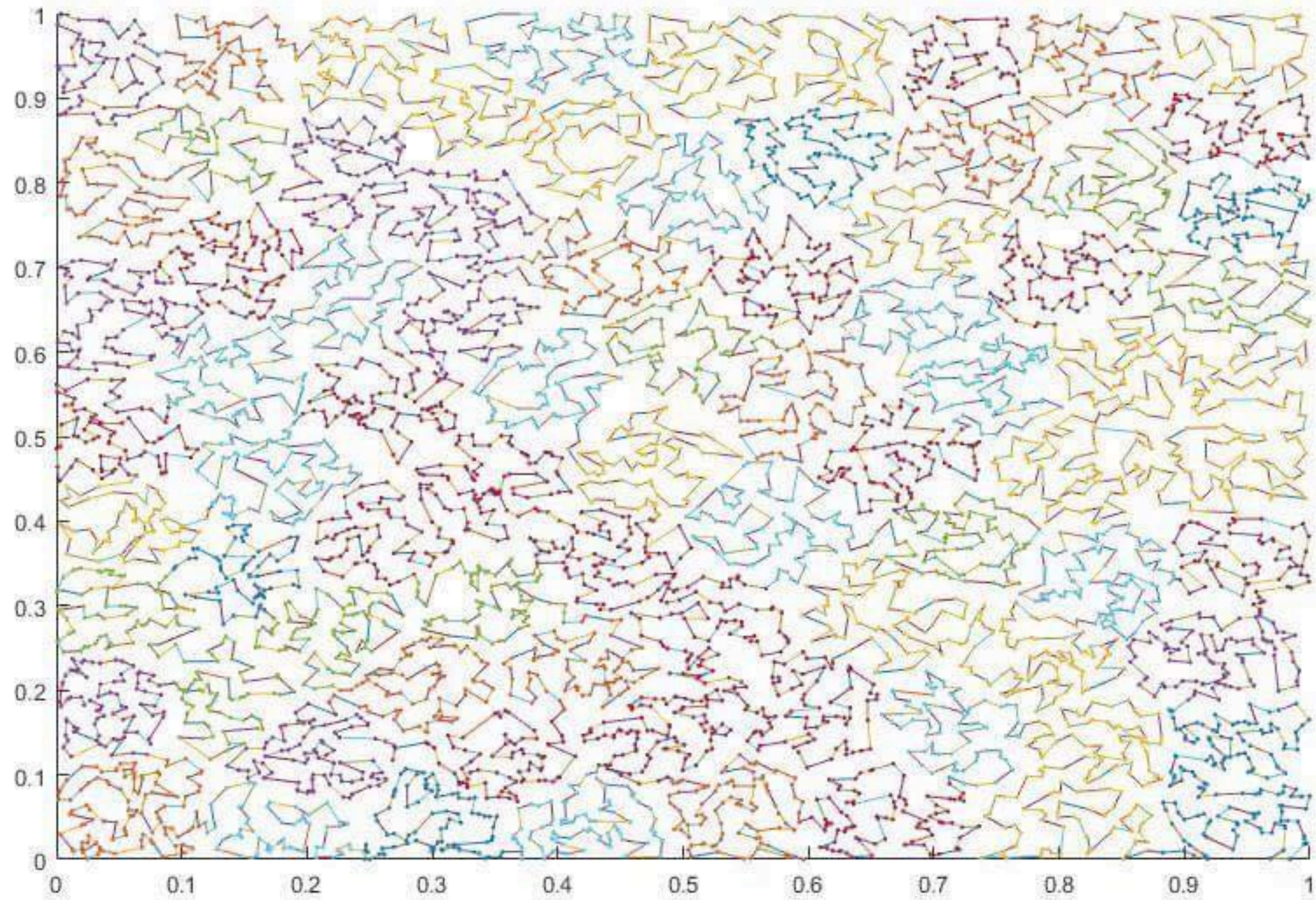
Examples of graphs

Graphs often arise in transportation and communication networks. Here's a (now very out-of-date) route map for Jet Blue airlines, originally taken from http://www.jetblue.com/travelinfo/routemap.html:





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- The **web graph** is a directed multigraph with web pages for vertices and hyperlinks for edges. Though it changes constantly, its properties have been fanatically studied both by academic graph theorists and employees of search engine companies, many of which are still in business.

- Companies like Google base their search rankings largely on structural properties of the web graph.

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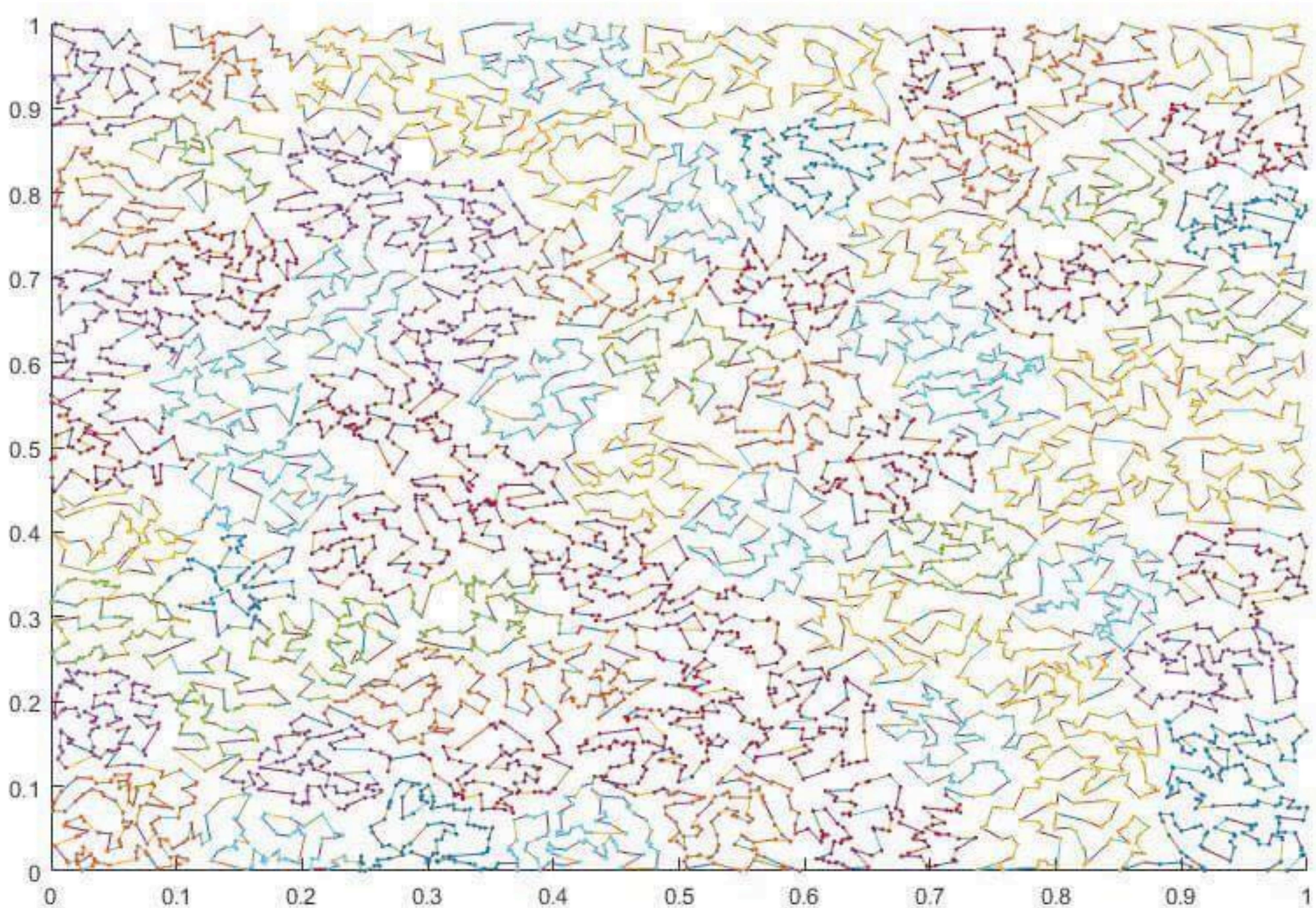
Traveling Salesman Problem



Local structure of graphs

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- Incidence: a vertex is **incident** to any edge of which it is an endpoint (and vice versa).

- Adjacency, neighborhood: two vertices are **adjacent** if they are the endpoints of some edge. The **neighborhood** of a vertex v is the set of all vertices that are adjacent to v.

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- Degree, in-degree, out-degree: the **degree** of v counts the number edges incident to v.
- In a directed graph, **in-degree** counts only incoming edges and **out-degree** counts only outgoing edges
  - (so that the degree is always the in-degree plus the out-degree).
- The degree of a vertex v is often abbreviated as d(v); in-degree and out-degree are similarly abbreviated as $d^{-}(v)$ and $d^{+}(v)$, respectively.
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Some standard graphs

- **Complete graph** K_n. This has n vertices, and every pair of vertices has an edge between them.
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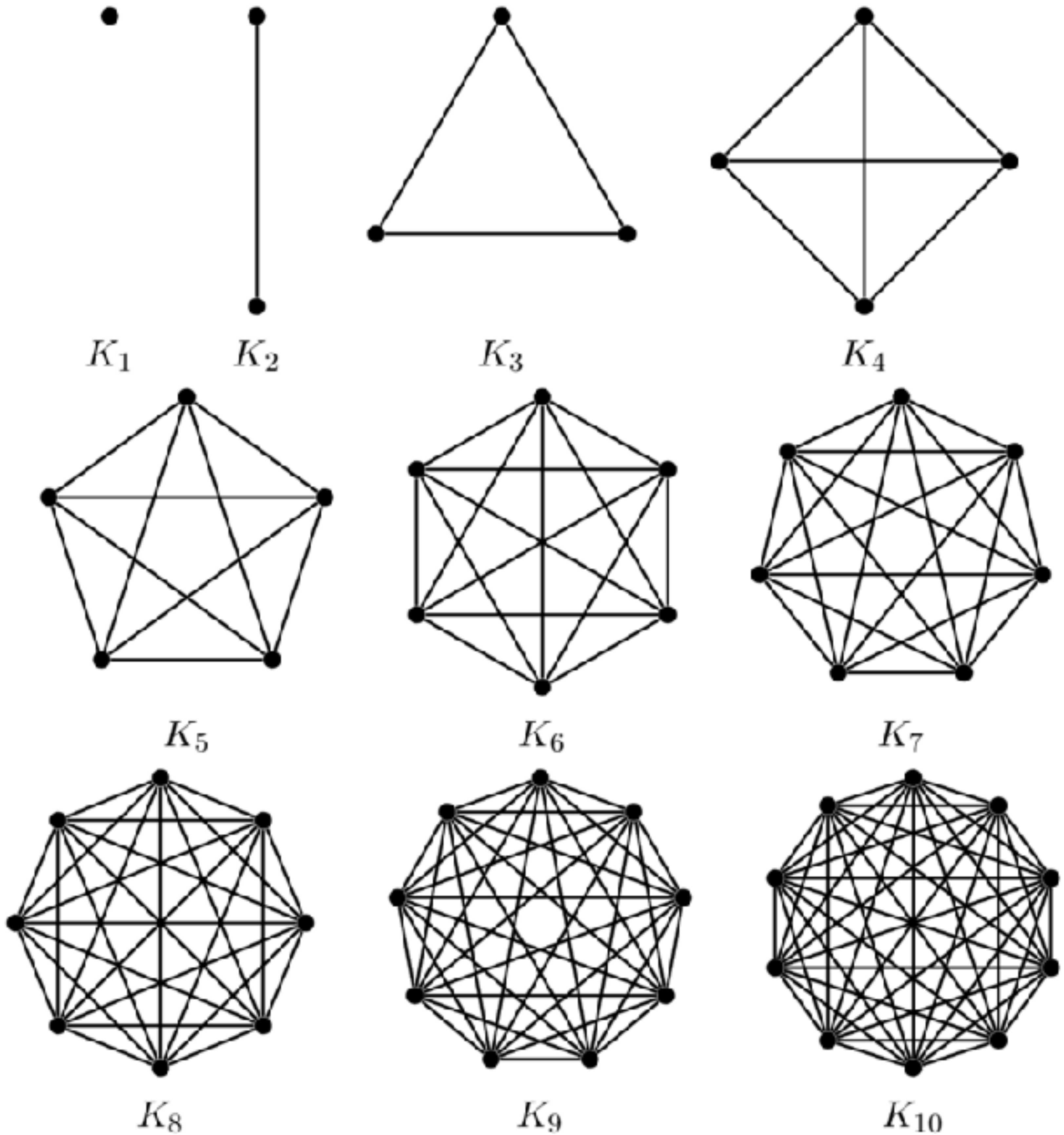

Figure 10.4: Complete graphs K_1 through K_{10}

- **Cycle** graph C_n. This has vertices {0,1,...n−1} and an edge from i to i+1 for each i, plus an edge from n−1 to 0. For any cycle, n must be at least 3. See Figure 10.5.
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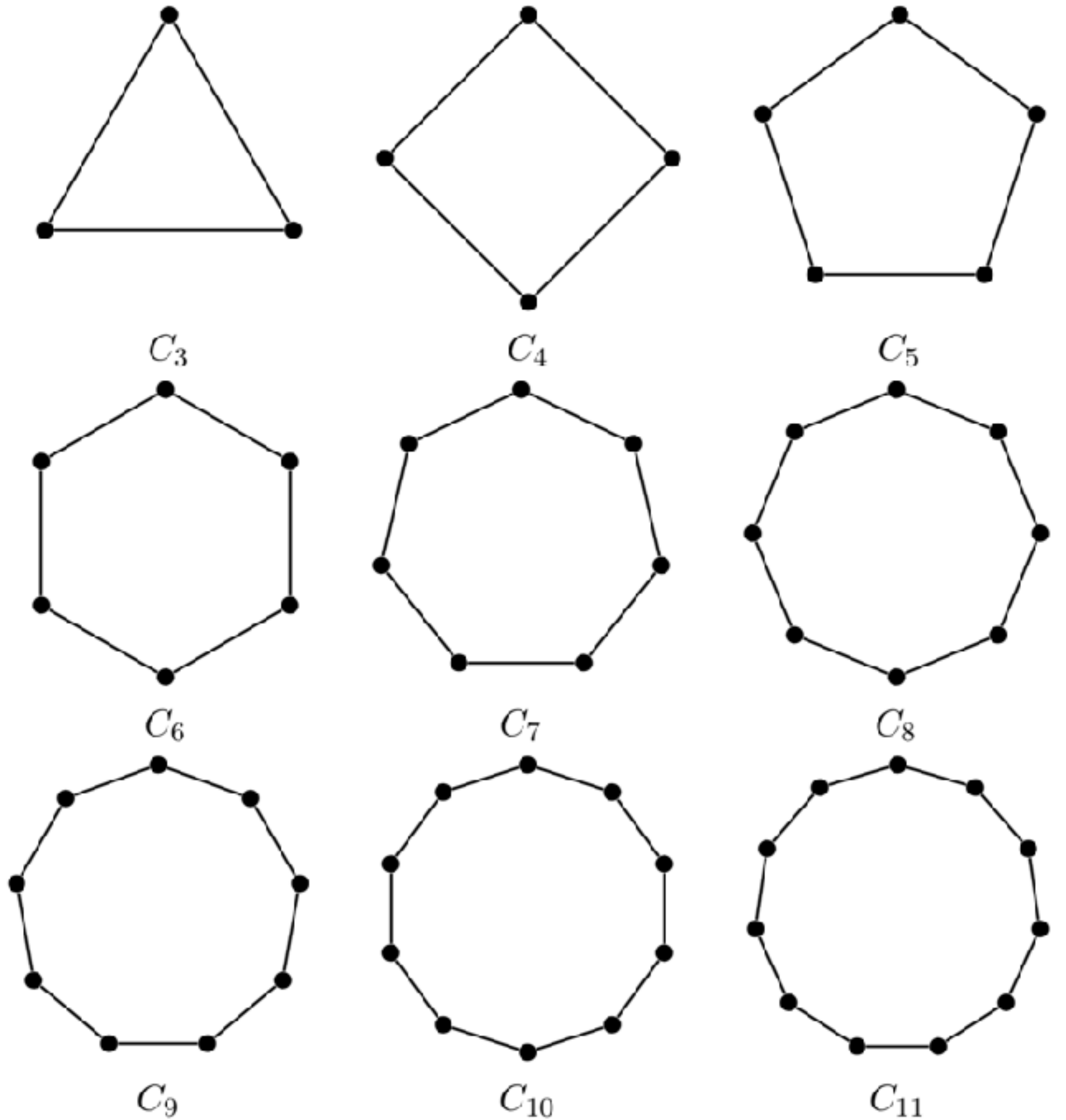


Figure 10.5: Cycle graphs C_3 through C_{11}

- **Cycle** graph C_n. This has vertices $\{0, 1, \ldots n-1\}$ and an edge from i to $i+1$ for each i, plus an edge from $n-1$ to 0. For any cycle, n must be at least 3. See Figure 10.5.

- **Path** P_n. This has vertices $\{0, 1, 2, \ldots n\}$ and an edge from i to $i+1$ for each i. Note that, despite the usual convention, n counts the number of *edges* rather than the number of vertices; we call the number of edges the **length** of the path. See Figure 10.6.

- **Complete bipartite graph** $K_{m,n}$. This has a set A of m vertices and a set B of n vertices, with an edge between every vertex in A and every vertex in B, but no edges within A or B. See Figure 10.7.

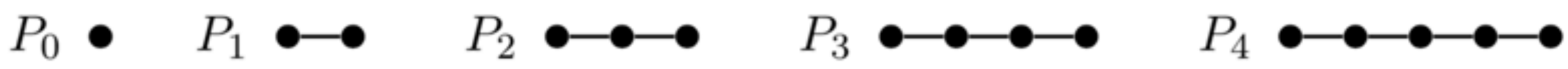
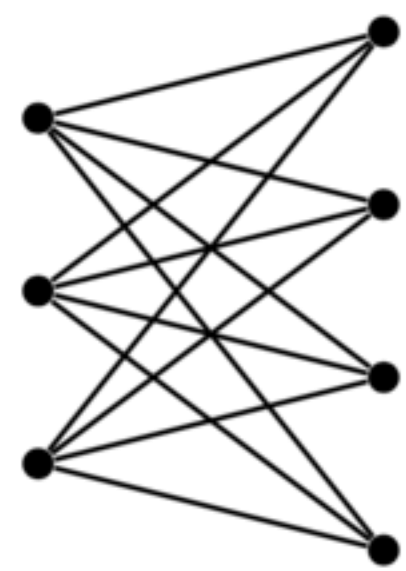


Figure 10.6: Path graphs P_0 through P_4



$K_{3,4}$

Figure 10.7: Complete bipartite graph $K_{3,4}$

- **Path** P_n. This has vertices {0,1,2,...n} and an edge from i to i+1 for each i. Note that, despite the usual convention, n counts the number of *edges* rather than the number of vertices; we call the number of edges the **length** of the path. See Figure 10.6.

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- **Complete bipartite graph** $K_{m,n}$. This has a set A of m vertices and a set B of n vertices, with an edge between every vertex in A and every vertex in B, but no edges within A or B. See Figure 10.7.
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- **Star graphs**. These have a single central vertex that is connected to n outer vertices, and are the same as $K_{1,n}$. See Figure 10.8.

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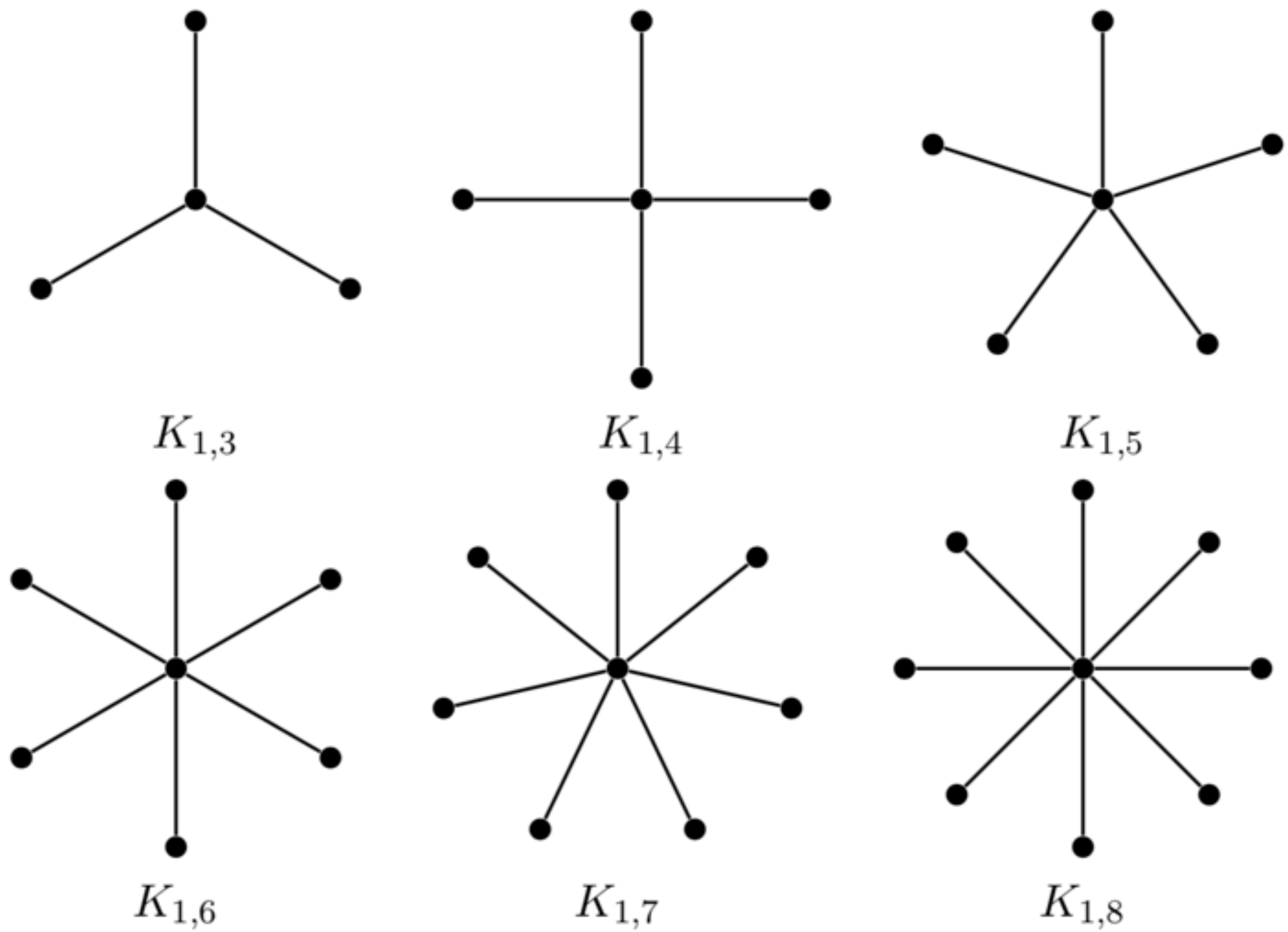

Figure 10.8: star graphs $K_{1,3}$ through $K_{1,8}$

- The **cube** Q_n. This is defined by letting the vertex set consist of all n-bit strings, and putting an edge between u and u′ if u and u′ differ in exactly one place. It can also be defined by taking the n-fold square product of an edge with itself (see §10.6).

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- **Cayley graphs**. The Cayley graph of a group G with a given set of generators S is a labeled directed graph.

- The vertices of this graph are the group elements, and for each element g in G and generator s in S there is a directed edge from g to gs labeled with s. An example of a small Cayley graph, based on the **dihedral group** D_4 of symmetries of the square, is given in Figure 10.9.

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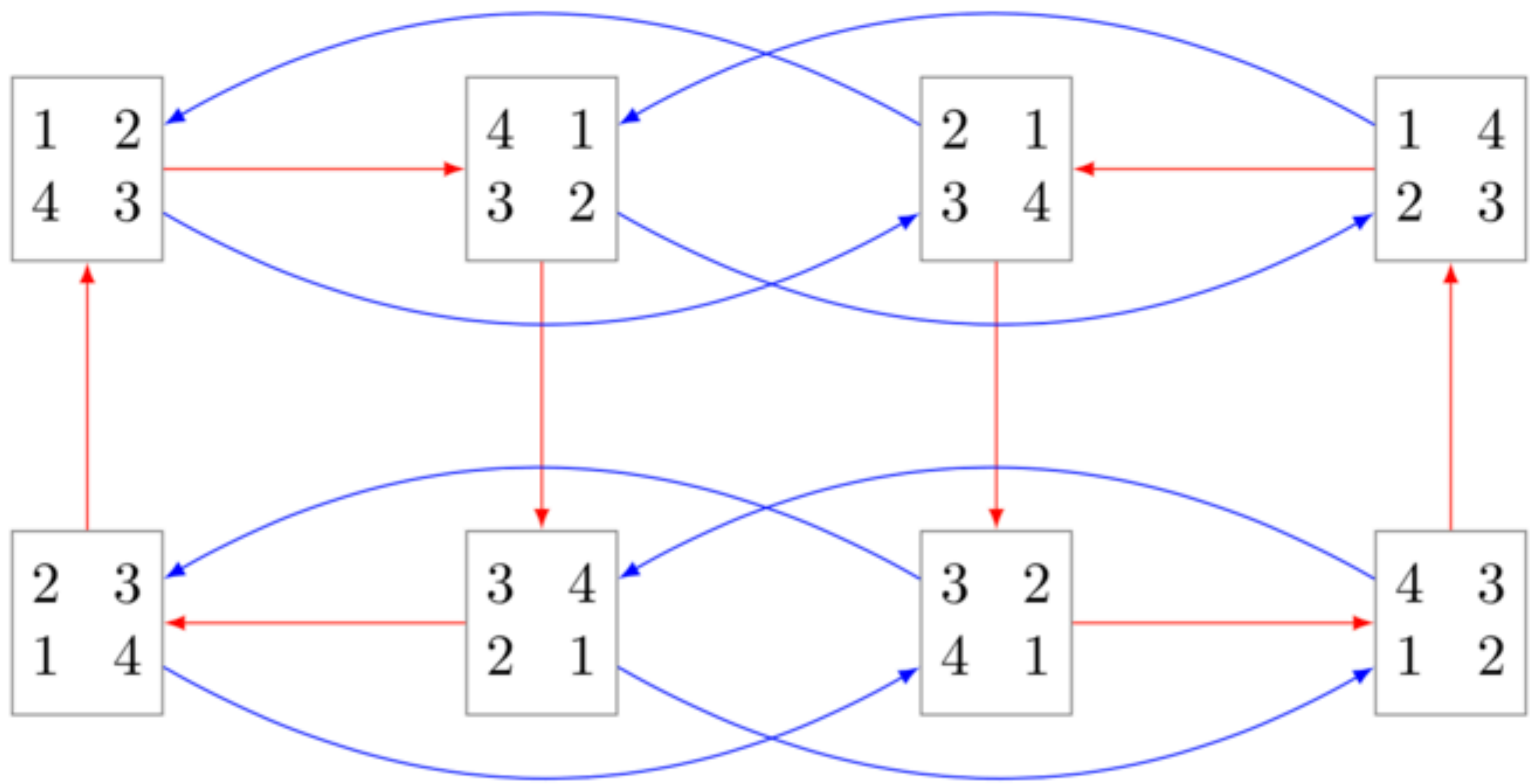

Figure 10.9: Cayley graph of the dihedral group D_4, with generators a corresponding to a clockwise rotation (red arrows) and b corresponding to a flip around the vertical axis (blue arrows). Note that this is a directed graph.

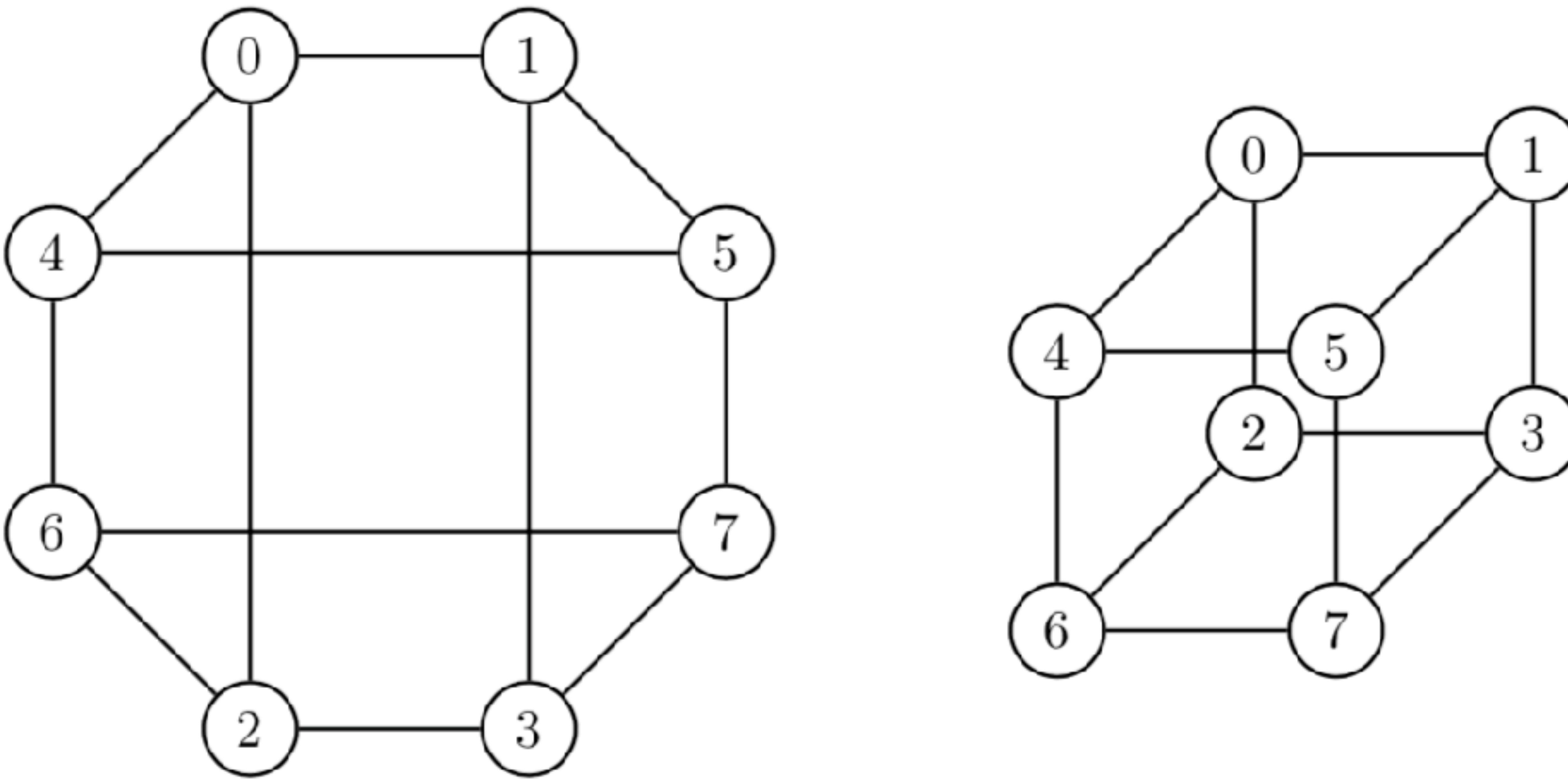

Figure 10.10: Two presentations of the cube graph Q_3

subgraph

- A graph G is a **subgraph** of a graph H, written $G \subseteq H$, if $V_G \subseteq V_H$ and $E_G \subseteq E_H$. We will also sometimes say that G is a subgraph of H if it is isomorphic to a subgraph of H, which is equivalent to having an injective homomorphism from G to H.

- One can get a subgraph by deleting edges or vertices or both. Note that deleting a vertex also requires deleting any edges incident to the vertex (since we can’t have an edge with a missing endpoint). If we delete as few edges as possible, we get an **induced subgraph**. Formally, the subgraph of a graph H whose vertex set is S and that contains every edge in H with endpoints in S is called the subgraph of H induced by S.
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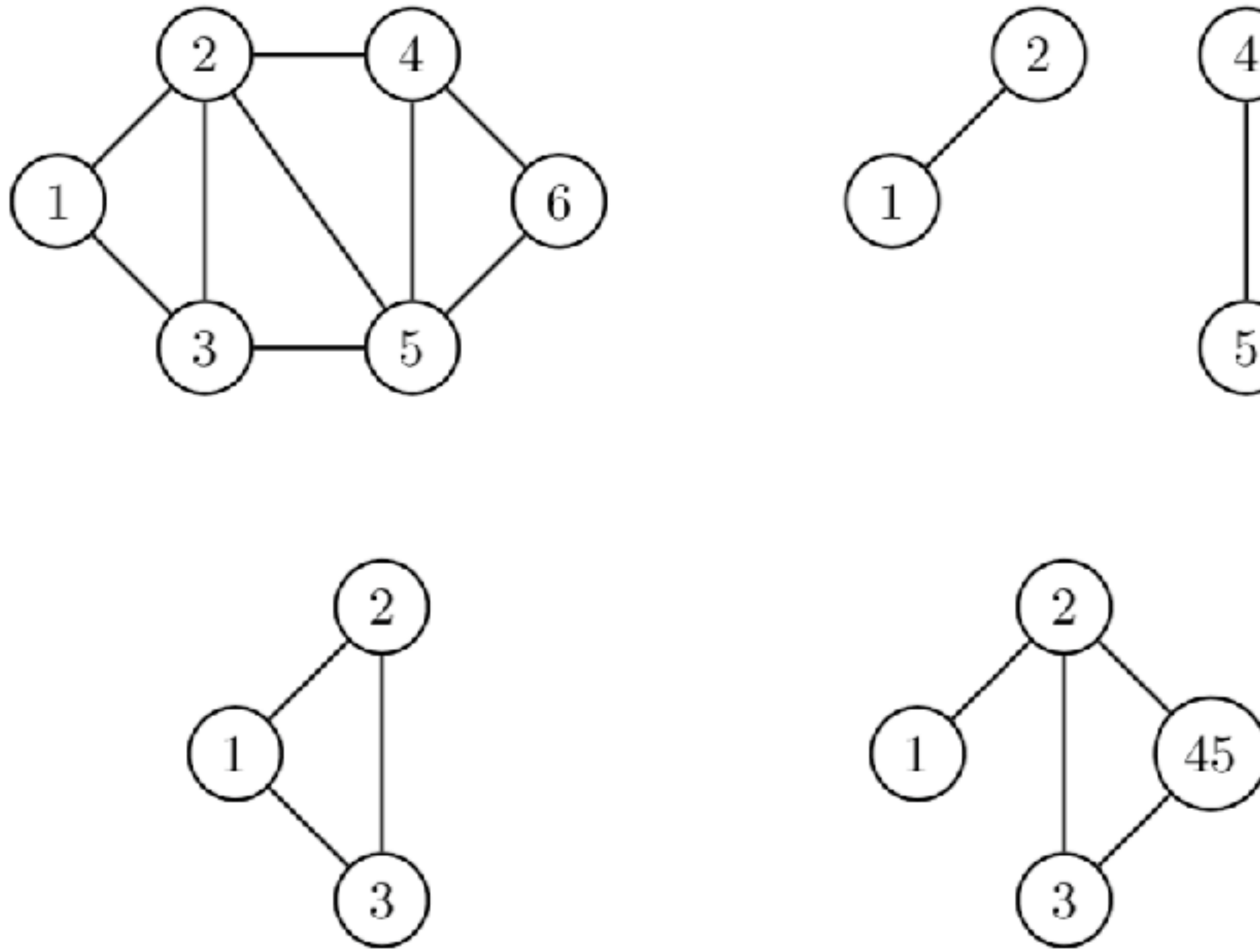

Figure 10.11: Examples of subgraphs and minors. Top left is the original graph. Top right is a subgraph that is not an induced subgraph. Bottom left is an induced subgraph. Bottom right is a minor.

- A **minor** of a graph H is a graph obtained from H by deleting edges and/or vertices (as in a subgraph) and **contracting** edges, where two adjacent vertices u and v are merged together into a single vertex that is adjacent to all of the previous neighbors of both vertices.

- Minors are useful for recognizing certain classes of graphs. For example, a graph can be drawn in the plane without any crossing edges if and only if it doesn’t contain K_5 or $K_{3,3}$ as a minor (this is known as **Wagner’s theorem**).

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