count I

Spanning trees

- A **spanning tree** of a nonempty connected graph G is a subgraph of G that includes all vertices and is a tree
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- **Theorem 10.10.9.** Every nonempty connected graph has a spanning tree.

- *Proof.* Let G = (V,E) be a nonempty connected graph.

- We'll show by induction on |E| that G has a spanning tree. The base case is |E| = |V | – 1 (the least value for which G can be connected); then G itself is a tree (by the theorem above).

- For larger |E|, the same theorem gives that G contains a cycle.

- Let uv be any edge on the cycle, and consider the graph G – uv; this graph is connected (since we can route any path that used to go through uv around the other edges of the cycle) and has fewer edges than G, so by the induction hypothesis there is some spanning tree T of G – uv. But then T also spans G, so we are done.

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Eulerian cycles

- **Theorem 10.10.10.** Let G be a connected graph. Then G has an Eulerian cycle if and only if all nodes have even degree.

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- (Only if part). Fix some cycle, and orient the edges by the direction that the cycle traverses them. Then in the resulting directed graph we must have $d^{-}(u) = d^{+}(u)$ for all u, since every time we enter a vertex we have to leave it again.

- But then $d(u) = 2d^{+}(u)$ is even.

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- Suppose now that d(u) is even for all u. We will construct an Eulerian cycle on all nodes by induction on |E|. The base case is when |E| = 2|V | and G = $C_{|V|}$.

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- For a larger graph, choose some starting node u_1, and construct a path $u_1u_2 . . .$ by choosing an arbitrary unused edge leaving each u_i; this is always possible for $u_i \neq u_1$ since whenever we reach u_i we have always consumed an even number of edges on previous visits plus one to get to it this time, leaving at least one remaining edge to leave on.

- Since there are only finitely many edges and we can only use each one once, eventually we must get stuck, and this must occur with $u_k = u_1$ for some k.

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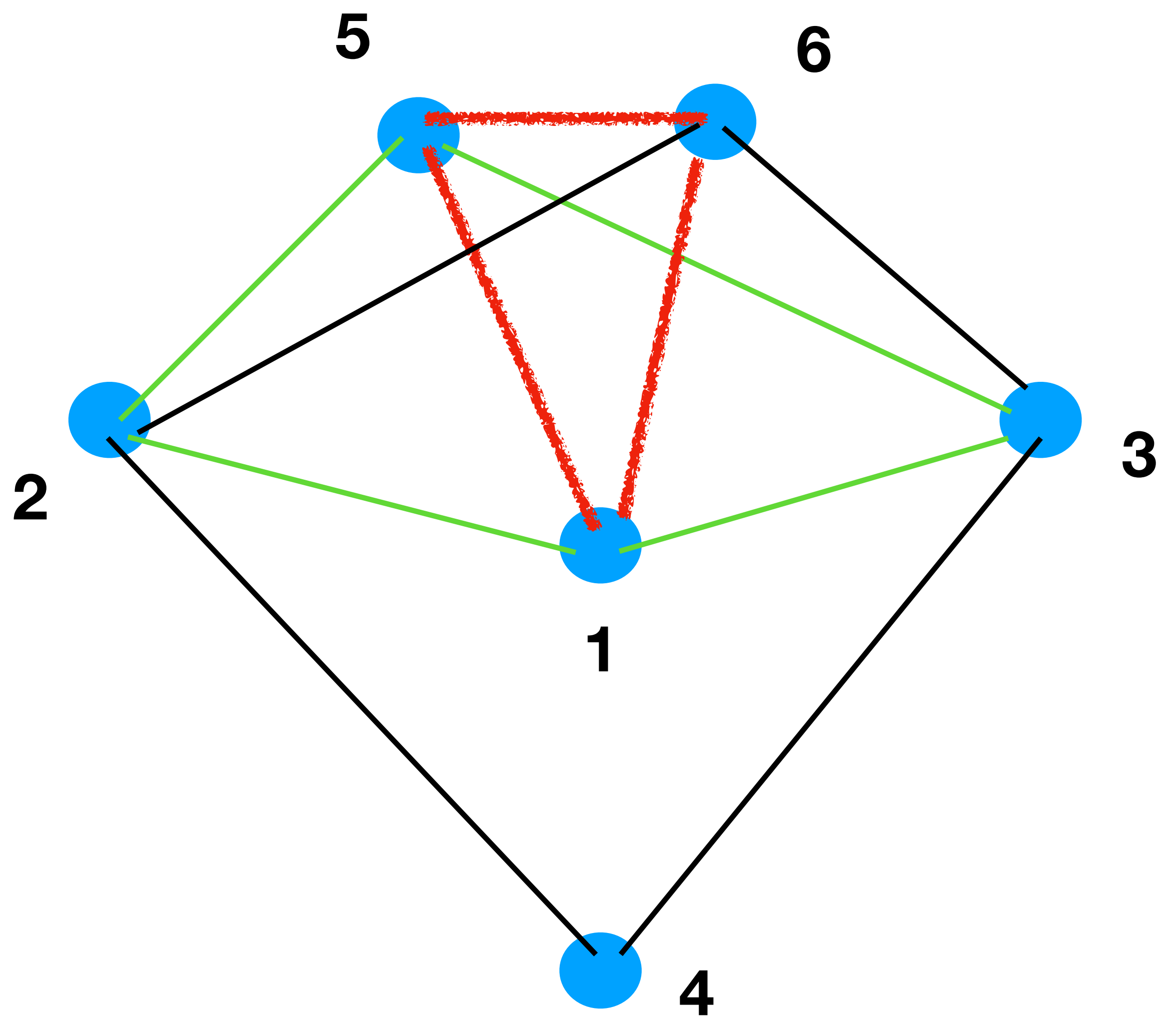

path : 1 5 6 1

path : 1 2 5 3 1

- Now delete all the edges in $u_1 \dots u_k$ from G, and consider the connected components of $G - (u_1 \dots u_k)$. Removing the cycle reduces d(v) by an even number, so within each such connected component the degree of all vertices is even. It follows from the induction hypothesis that each connected component has an Eulerian cycle.

- We'll now string these per-component cycles together using our original cycle: while traversing $u_1 \dots, u_k$ when we encounter some component for the first time, we take a detour around the component's cycle. The resulting merged cycle gives an Eulerian cycle for the entire graph.

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- Why doesn't this work for Hamiltonian cycles? The problem is that in a Hamiltonian cycle we have too many choices: out of the d(u) edges incident to u, we will only use two of them. If we pick the wrong two early on, this may prevent us from ever fitting u into a Hamiltonian cycle. So we would need some stronger property of our graph to get Hamiltonicity.

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Counting

- Counting is the process of creating a bijection between a set we want to count and some set whose size we already know. Typically this second set will be a finite ordinal [n] = {0, 1, . . . , n – 1}.

- Counting a set A using a bijection f : A → [n] gives its size |A| = n; this size is called the **cardinality** of n.

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- As a side effect, it also gives a well-ordering of A, since [n] is well-ordered as we can define $x \leq y$ for x, y in A by $x \leq y$ if and only if $f(x) \leq f(y)$.

- Often the quickest way to find f is to line up all the elements of A in a well-ordering and then count them off: the smallest element of A gets mapped to 0, the next smallest to 1, and so on.

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enumerative combinatorics

- standard counting principles based on how we constructed the set
- The branch of mathematics that studies sets constructed by combining other sets is called **combinatorics**
- the sub-branch that counts these sets is called **enumerative combinatorics**
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- For infinite sets, cardinality is a little more complicated. The basic idea is that we define |A| = |B| if there is a bijection between them.

- This gives an equivalence relation on sets[2], and we define |A| to be the equivalence class of this equivalence relation that contains A.

Basic counting techniques

- the number of subsets of a set of size n,
- the number of ways to put k cats into n boxes so that no box gets more than one cat
- theset$S_n = x \in N x < n^2 \wedge \exists y : x = y^2$hasexactlyn members, because we can generate it by applying the one-to-one correspon- dencef$(y) = y^2$totheset$\{0,1,2,3,...,n-1\} = [n]$
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- Constructing an explicit one-to-one correspondence is too time-consuming or too hard, so instead we will show how to

- map set-theoretic operations to arithmetic operations, so that from a set-theoretic construction of a set we can often directly read off an arithmetic computation that gives the size of the set.

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- Equality: reducing to a previously-solved case
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- Inequalities: showing $|A| \leq |B|$ and $|B| \leq |A|$

- We write $|A| \leq |B|$ if there is an injection $f : A \to B$, and similarly $|B| \leq |A|$ if there is an injection $g : B \to A$. If both conditions hold, then there is a bijection between A and B, showing $|A| = |B|$. This fact is trivial for finite sets, but for infinite sets—even though it is still true—the actual construction of the bijection is a little trickier.[3]

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- Similarly, if we write $|A| \geq |B|$ to indicate that there is a surjection from A to B, then $|A| \geq |B|$ and $|B| \geq |A|$ implies $|A| = |B|$. The easiest way to show this is to observe that if there is a surjection $f : A \to B$, then we can get an injection $f' : B \to A$ by letting $f'(y)$ be any element of $\{x \mid f(x) = y\}$, thus reducing to the previous case

- Showing an injection $f : A \to B$ and a surjection $g : A \to B$ also works.

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- For example, |Q| = |N|.

- Proof: |N| ≤ |Q| because we can map any n in N to the same value in Q; this is clearly an injection.

- To show |Q| ≤ |N|, observe that we can encode any element ±p/q of Q, where p and q are both natural numbers, as a triple (s, p, q) where (s ∈ {0, 1} indicates + (0) or – (1); this encoding is clearly injective.

- Then use the Cantor pairing function (§3.7.1) twice to crunch this triple down to a single natural number, getting an injection from Q to N.

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Addition: the sum rule

- The **sum rule** computes the size of $A \cup B$ when A and B are disjoint. **Theorem 11.1.1.** *If* A *and* B *are finite sets with* $A \cap B = \varnothing$*, then* $|A \cup B| = |A| + |B|$.

Proof. Let $f : A \to [|A|]$ and $g : B \to [|B|]$ be bijections.

Define $h : A \cup B \to [|A| + |B|]$ by the rule $h(x) = f(x)$ for $x \in A$, $h(x) = |A| + g(x)$ for $x \in B$.

To show that this is a bijection, define

$h^{-1}(y)$ for y in $[|A| + |B|]$ to be $f^{-1}(y)$ if $y < |A|$ and $g^{-1}(y - |A|)$ otherwise.

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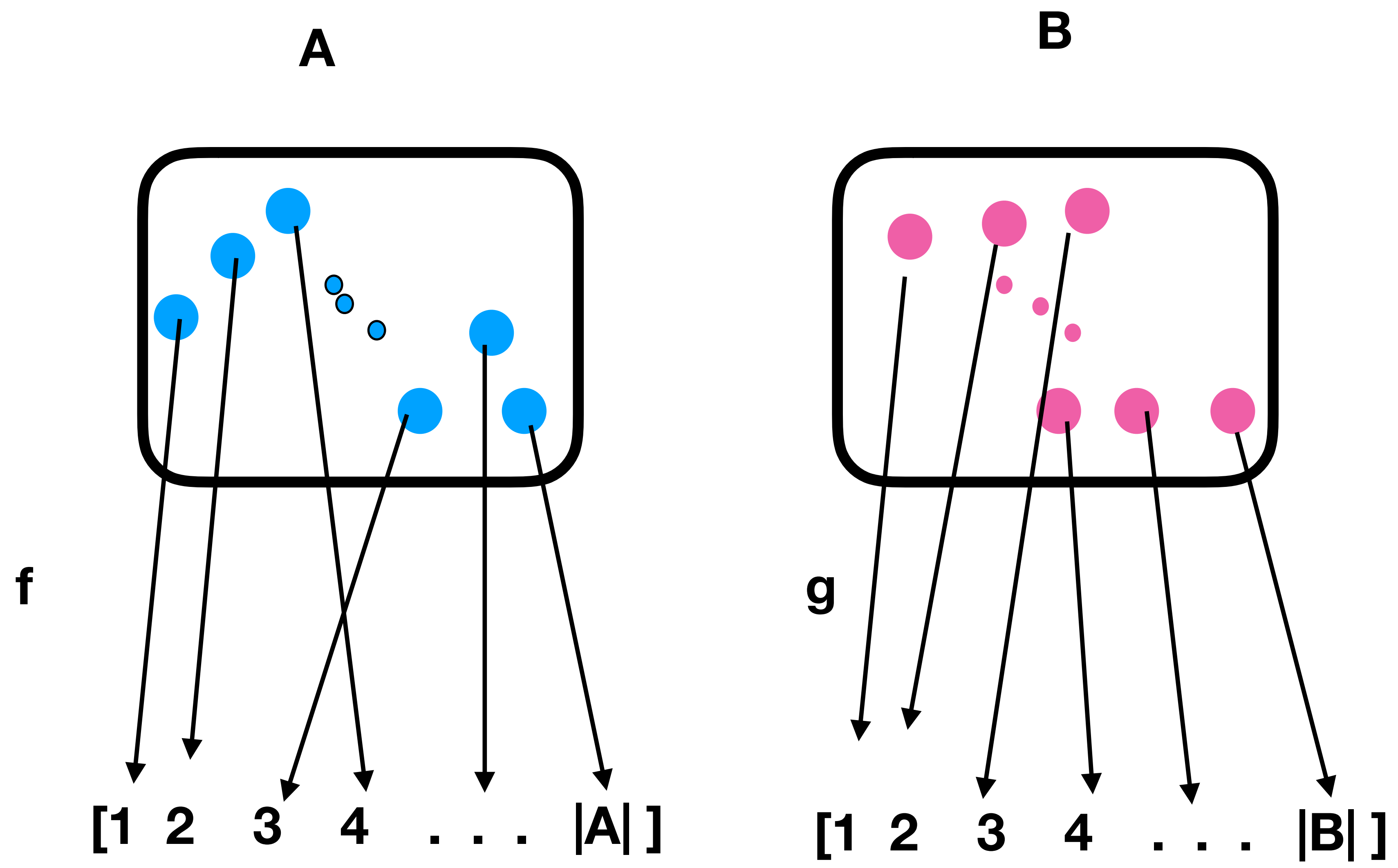
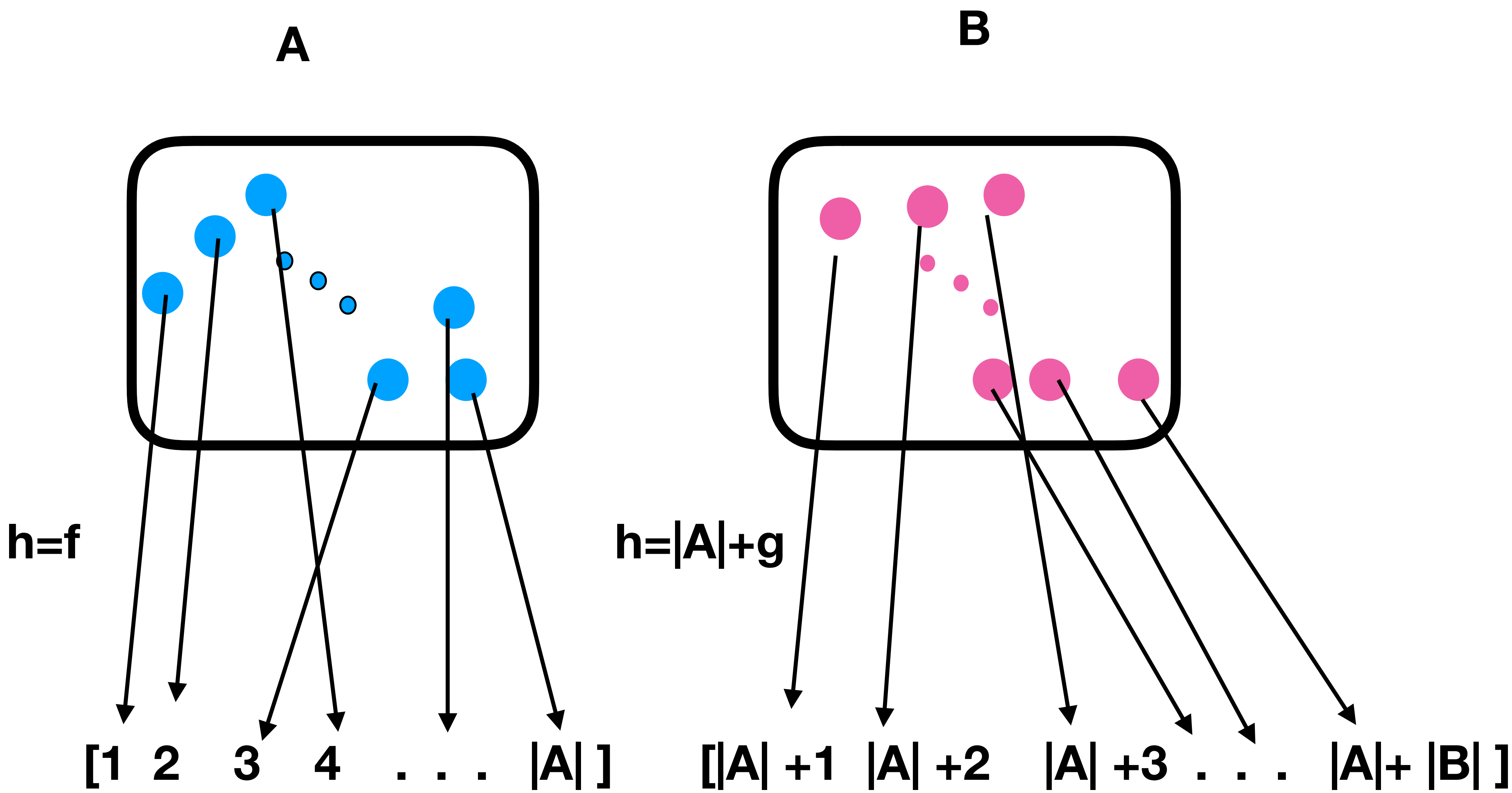

A
B
f
g
[1 2 3 4 . . . |A|]
[1 2 3 4 . . . |B|]



$h^{-1} = ?$

Then for any y in $[|A| + |B|]$, either

- $0 \leq y < |A|$, y is in the codomain of f
  - $h^{-1}(y) = f^{-1}(y) \in A$ is well-defined, and $h(h^{-1}(y)) = f(f^{-1}(y)) = y$.

- $|A| \leq y < |A|+|B|$. In this case $0 \leq y-|A| < |B|$, putting $y-|A|$ in the codomain of g and giving $h(h^{-1}(y)) = g(g^{-1}(y - |A|)) + |A| = y$.

So h^{-1} is in fact an inverse of h, meaning that h is a bijection.

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Generalizations: If $A_1, A_2, A_3 \dots A_k$ are **pairwise disjoint** (i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$), then

$$\left| \bigcup_{i=1}^{k} A_i \right| = \sum_{i=1}^{k} |A_i|.$$

The proof is by induction on k.

Example: As I was going to Saint Ives, I met a man with 7 wives, 28 children, 56 grandchildren, and 122 great-grandchildren. Assuming these sets do not overlap, how many people did I meet? Answer: 1+7+28+56+122=214.

For infinite sets

- he sum rule works for infinite sets, too; technically, the sum rule is used to *define* $|A| + |B|$ as $|A \cup B|$ when A and B are disjoint. This makes cardinal arithmetic a bit wonky: if at least one of A and B is infinite, then $|A| + |B| = \max(|A|, |B|)$, since we can space out the elements of the larger of A and B and shove the elements of the other into the gaps.
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The Pigeonhole Principle

- A consequence of the sum rule is that if A and B are both finite and $|A| > |B|$, you can't have an injection from A to B.
- The proof is by contraposition. prove $p \rightarrow q$ by verifying $\sim q \rightarrow \sim p$
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- Suppose f : A → B is an injection.

- Write A as the union of $f^{-1}(x)$ for each $x \in B$, where $f^{-1}(x)$ is the set of y in A that map to x. Because each $f^{-1}(x)$ is disjoint, the sum rule applies; but because f is an injection there is at most one element in each $f^{-1}(x)$.

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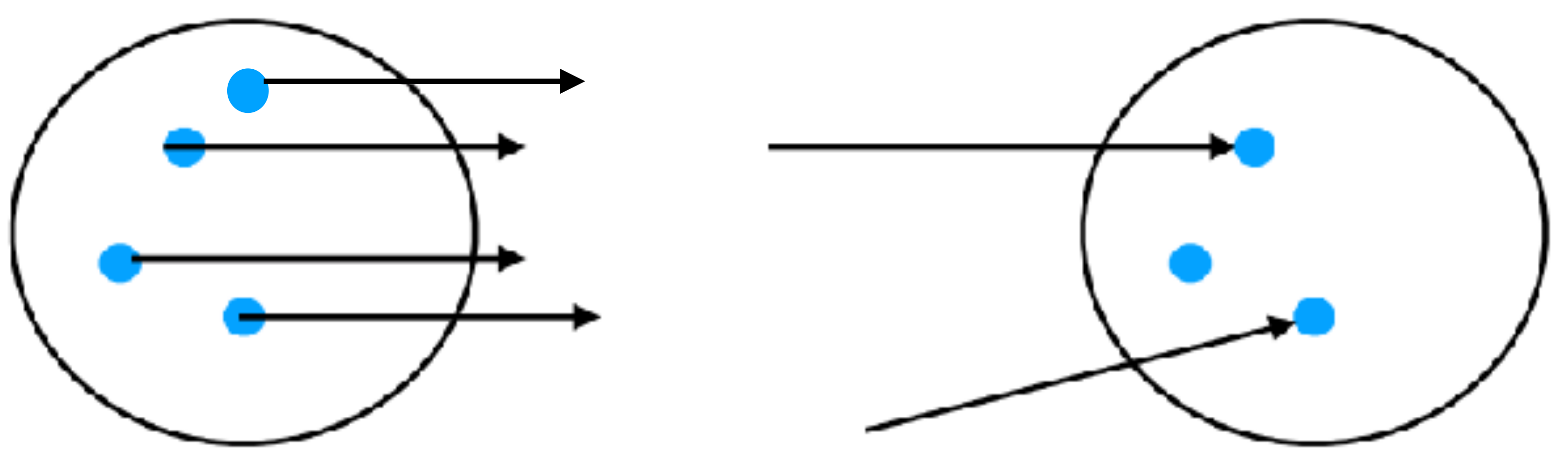

- It follows that $|A| = \sum_{x \in B} f^{-1}(x) \leq \sum_{x \in B} 1 = |B|$.

- # Pigeonhole

- If we have n boxes and we place more than n objects into them, then there will be at least one box that contains more than one object.

$$|A| > |B|$$

The Pigeonhole Principle generalizes in an obvious way to functions with larger domains; if $f : A \to B$, then there is some x in B such that $\left|f^{-1}(x)\right| \geq |A|/|B|$.

Subtraction

- For any sets A and B, A is the disjoint union of $A \cap B$ and $A \setminus B$.
- So $|A| = |A \cap B| + |A \setminus B|$ (for finite sets) by the sum rule.
- Rearranging gives $|A \setminus B| = |A| - |A \cap B|$. (11.1.1)
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Theorem 11.1.2. *For any finite sets A and B,*

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Proof. Compute

$$\begin{aligned} |A \cup B| &= |A \cap B| + |A \setminus B| + |B \setminus A| \\ &= |A \cap B| + (|A| - |A \cap B|) + (|B| - |A \cap B|) \\ &= |A| + |B| - |A \cap B|. \end{aligned}$$

This is a special case of the **inclusion-exclusion formula**, which can be used to compute the size of the union of many sets using the size of pairwise, triple-wise, etc. intersections of the sets.

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Inclusion-exclusion for infinite sets

- Subtraction doesn't work very well for infinite quantities (while $\aleph_0 + \aleph_0 = \aleph_0$ that doesn't mean $\aleph_0 = 0$).
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- So the closest we can get to the inclusion- exclusion formula is that $|A|+|B|=|A \cup B|+|A \cap B|$.

- If at least one of A or B is infinite, then $|A \cup B|$ is also infinite, and since $|A \cap B| \leq |A \cup B|$ we have $|A \cup B| + |A \cap B| = |A \cup B|$ by the bizarre rules of cardinal arithmetic.

- So for infinite sets we have the rather odd result that $|A \cup B| = |A| + |B| = \max(|A|,|B|)$ whether the sets overlap or not.

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Combinatorial proof

- We can prove $|A| + |B| = |A \cup B| + |A \cap B|$ combinatorially, by turning both sides of the equation into disjoint unions (so the sum rule works) and then providing an explicit bijection between the resulting sets.
- The trick is that we can always force a union to be disjoint by tagging the elements with extra information;
  - so on the left-hand side we construct $L = \{0\} \times A \cup \{1\} \times B$
  - on the right-hand side we construct $R = \{0\}\times(A\cup B)\cup\{1\}\times(A\cap B)$.
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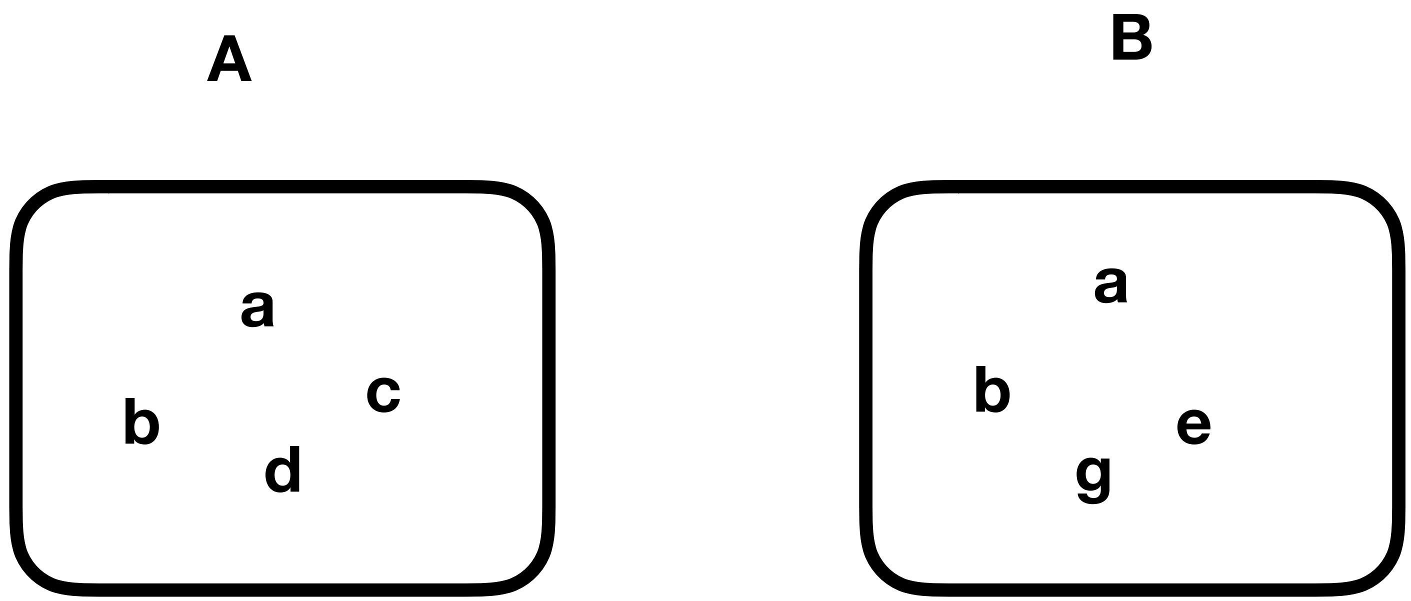
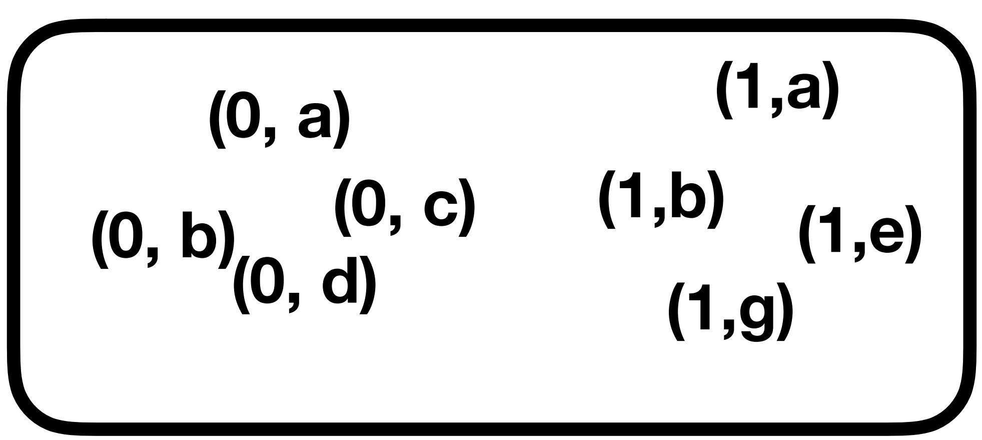
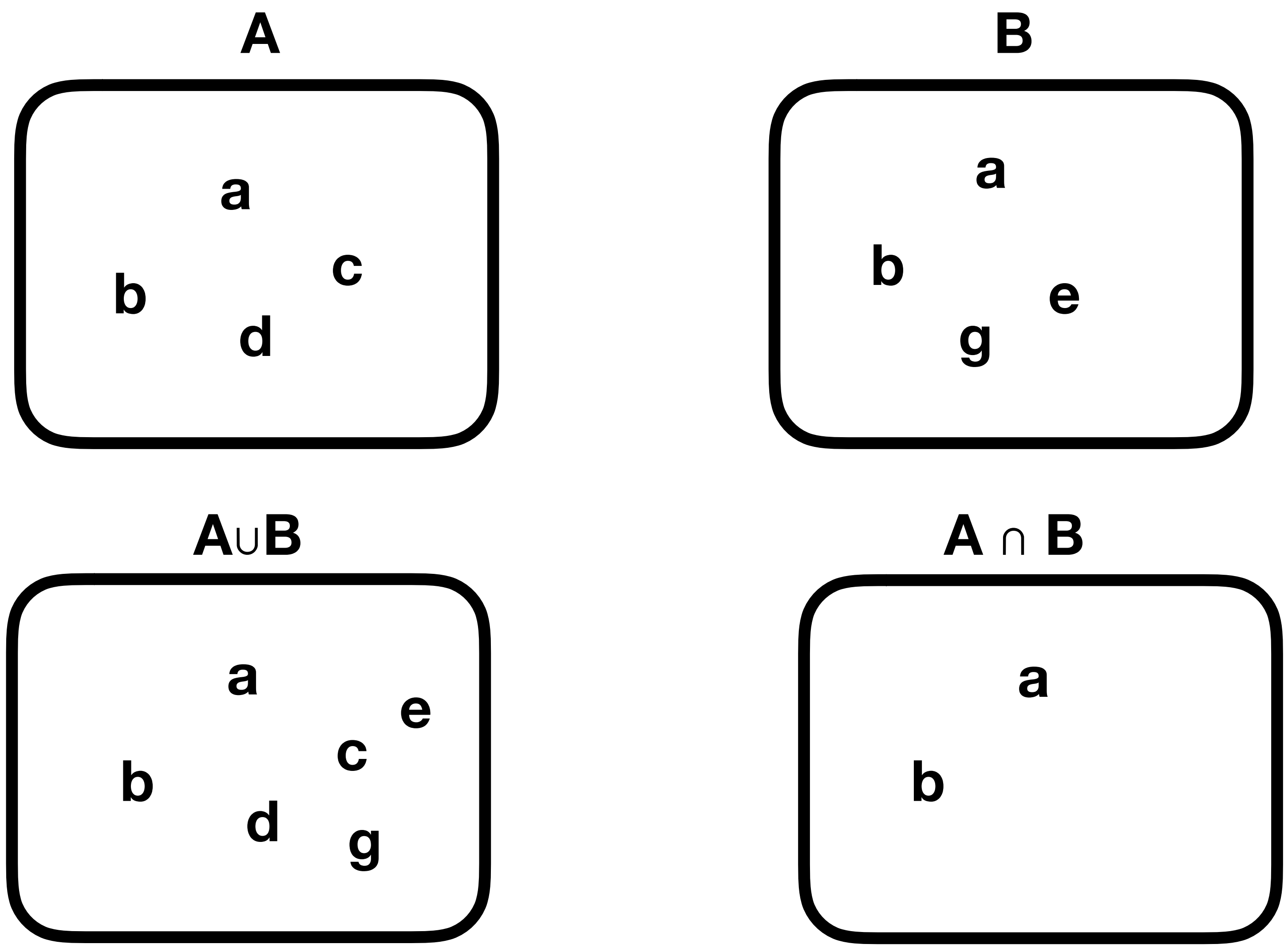
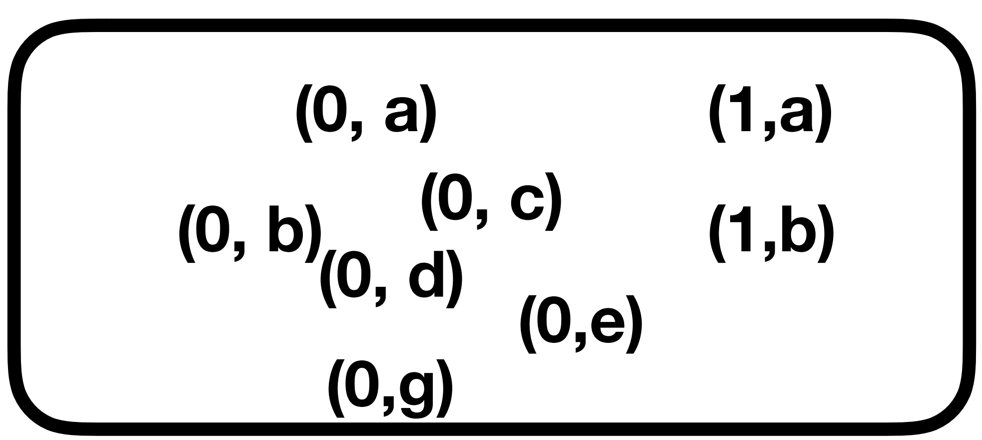


- $L = \{0\} \times A \cup \{1\} \times B$
- $|L| = |A|+|B|$





- $R = \{0\}\times(A\cup B)\cup\{1\}\times(A\cap B)$



- $|R| = |A\cup B|+|A\cap B|$

- It is easy to see that both unions are disjoint, because we are always taking the union of a set of ordered pairs that start with 0 with a set of ordered pairs that start with 1, and no ordered pair can start with both tags;

- it follows that $|L| = |A|+|B|$ and $|R| = |A \cup B|+|A \cap B|$.

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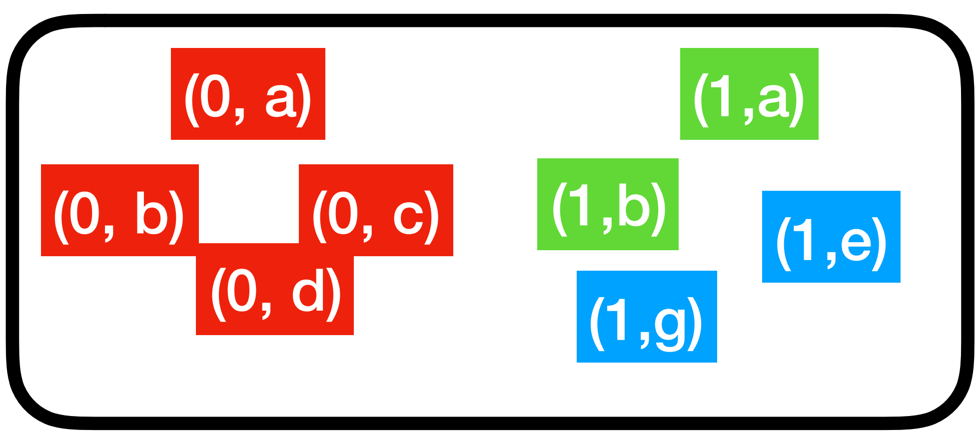
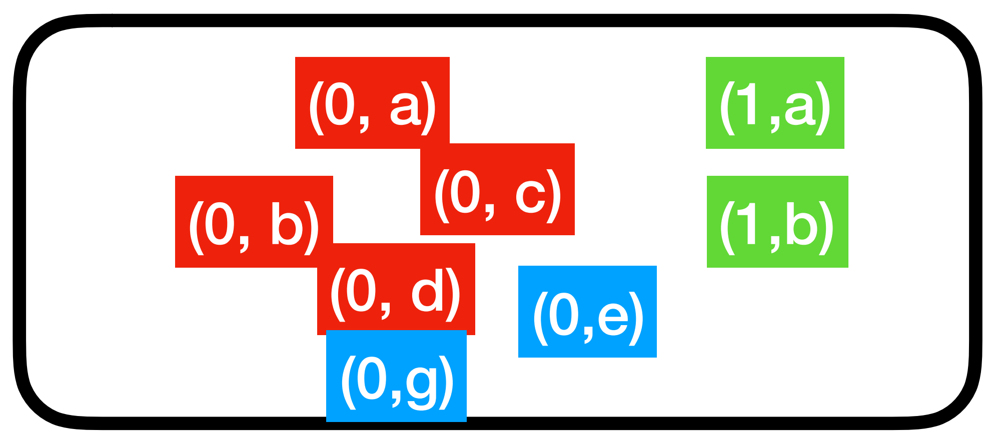
- Now define the function $f : L \to R$ by the rule
- $f((0,x)) = (0,x)$.
- $f((1, x)) = (1, x)$ if $x \in B \cap A$.
- $f((1, x)) = (0, x)$ if $x \in B \setminus A$.



- $L = \{0\} \times A \cup \{1\} \times B$

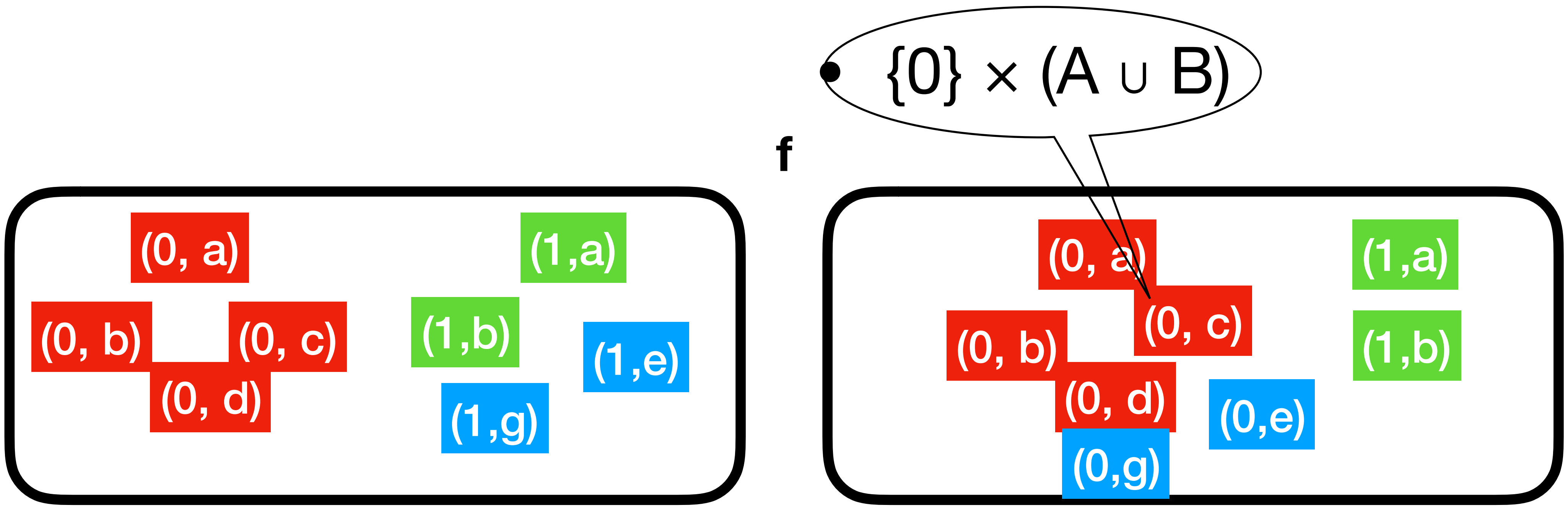


- $|R| = |A \cup B| + |A \cap B|$

Observe that f is surjective, because

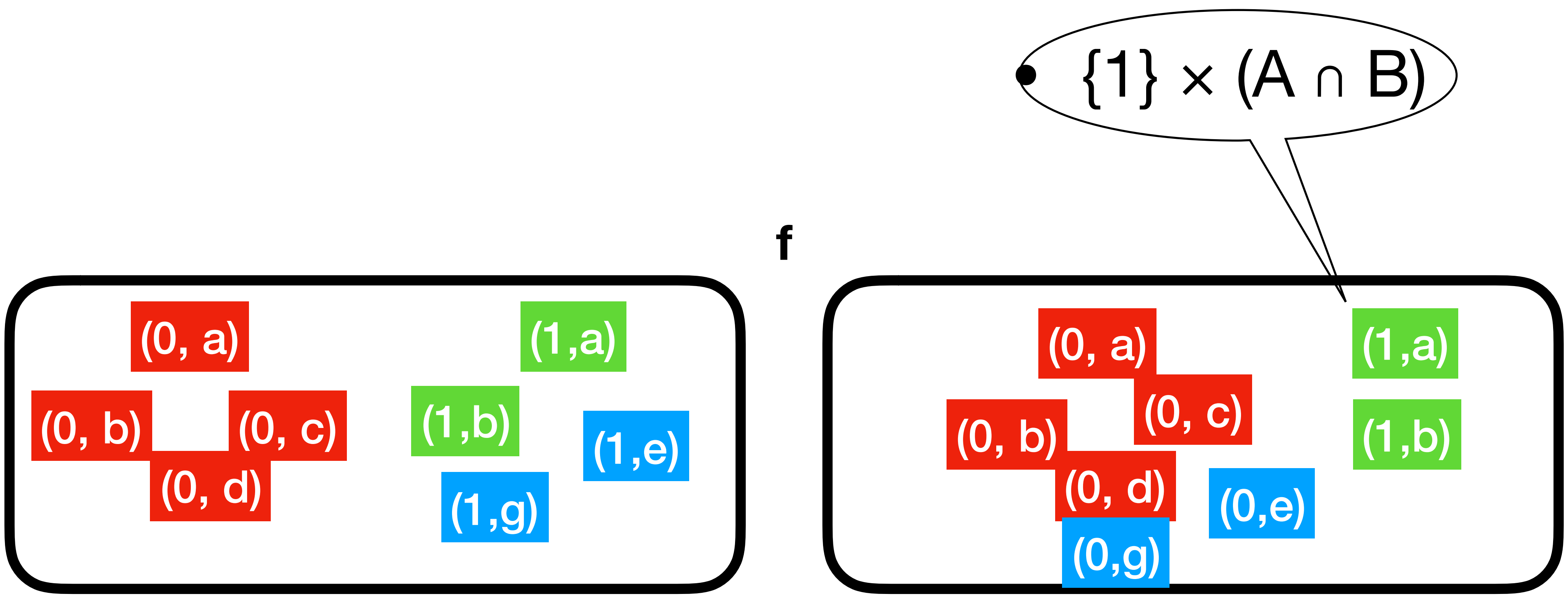
- for any $(0, x)$ in $\{0\} \times (A \cup B)$, either x is in A and $(0, x) = f((0, x))$ where $(0, x) \in L$,

- or x is in $B \setminus A$ and $(0, x) = f((1, x))$ where $(1, x) \in L$.



- $L = \{0\} \times A \cup \{1\} \times B$
- $|R| = |A \cup B| + |A \cap B|$

- for any (1, x) in {1} × (A ∩ B), x is in A ∩ B and (1, x) = f ((1, x)) where (1, x) ∈ L,



- L = {0} × A ∪ {1} × B
- |R| = |A∪B|+|A∩B|

- It is also true that f is injective; the only way for it not to be is if $f((0, x)) = f((1, x)) = (0, x)$ for some x.
- Suppose this occurs.
  - Then $x \in A$ (because of the 0 tag) and $x \in B \setminus A$ (because $(1,x)$ is only mapped to $(0,x)$ if $x \in B \setminus A$). But x can't be in both A and $B \setminus A$, so we get a contradiction.
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