count II

Multiplication: the product rule

- The **product rule** says that Cartesian product maps to arithmetic product.

- Intuitively, we line the elements (a, b) of $A \times B$ in lexicographic order and count them off.
  字典

- This looks very much like packing a two-dimensional array in a one-dimensional array by mapping each pair of indices (i, j) to $i \cdot |B| + j$

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Relate C(i,j) to D(k)

- C is an mxn array
- D(k) is related to C(i,j)
- for i ≥ 0 , $j \geq 0$
  - k= i*n+j

|B|=n=4

	0	1	2	3
0	D(0)	D(1)	D(2)	D(3)
1	D(4)			
2				
3				
4				D(19)

|A|=m=5

- $|A \times B| = |A| \cdot |B|$.

- *Proof.* The trick is to order $A \times B$ lexicographically and then count off the elements. Given bijections $f : A \to [|A|]$ and $g : B \to [|B|]$, define $h : (A \times B) \to [|A| \cdot |B|]$ by the rule $h((a,b)) = a \cdot |B| + b$.

- The division algorithm recovers a and b from h(a, b) by recovering the unique natural numbers q and r such that $h(a, b) = q \cdot |B| + r$ and $0 \leq b < |B|$ and letting $a = f^{-1}(q)$ and $b = g^{-1}(r)$.

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Relate D(k) to C(i,j)

- C is an mxn array
- Relate D(k) to C(i,j)
- for k ≥ 0 ,
  - j= mod(k,n)
  - i= (k-j)/n

|B|=n=4

	0	1	2	3
0	D(0)	D(1)	D(2)	D(3)
1	D(4)			
2				
3				
4				D(19)

|A|=m=5

The general form is

$$\left|\prod_{i=1}^{k} A_i\right| = \prod_{i=1}^{k} |A_i|,$$

where the product on the left is a Cartesian product and the product o
the right is an ordinary integer product.

- As I was going to Saint Ives, I met a man with seven sacks, and every sack had seven cats. How many cats total?

大布袋

- Answer: Label the sacks 0,1,2,...,6, and label the cats in each sack 0,1,2,...,6. Then each cat can be specified uniquely by giving a pair (sack number, cat number), giving a bijection between the set of cats and the set 7×7. Since $|7 \times 7| = 7 \cdot 7 = 49$, we have 49 cats.

科學怪人

- Dr. Frankenstein's trusty assistant Igor has brought him 6 torsos, 4 brains, 8 pairs of matching arms, and 4 pairs of legs. How many different monsters can Dr Frankenstein build?

- Answer: there is a one- to-one correspondence between possible monsters and 4-tuples of the form (torso, brain, pair of arms, pair of legs); the set of such 4-tuples has $6 \cdot 4 \cdot 8 \cdot 4 = 728$ members.

order or sorting

How many different ways can you order n items? Call this quantity $n!$ (pronounced "n **factorial**"). With 0 or 1 items, there is only one way; so we have $0! = 1! = 1$. For $n > 1$, there are n choices for the first item, leaving $n - 1$ items to be ordered. From the product rule we thus have $n! = n \cdot (n-1)!$, which we can expand out as $\prod_{i=1}^{n} i$, our previous definition of $n!$.

For infinite sets

- The product rule also works for infinite sets, because we again use it as a definition: for any A and B, $|A| \cdot |B|$ is defined to be $|A \times B|$

- One oddity for infinite sets is that this definition gives $|A| \cdot |B| = |A| + |B| = \max(|A|, |B|)$, because if at least one of A and B is infinite, it is possible to construct a bijection between $A \times B$ and the larger of A and B. Infinite sets are strange.

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Exponentiation: the exponent rule

Given sets A and B, let A^B be the set of functions $f : B \to A$. The
$\left|A^B\right| = |A|^{|B|}$.

- If |B| is finite, this is just a |B|-fold application of the product rule: we can write any function f : B → A as a sequence of length |B| that gives the value in A for each input in B. Since each element of the sequence contributes |A| possible choices, we get $|A|^{|B|}$ choices total.

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For infinite sets

- For infinite sets, the exponent rule is a definition of $|A|^{|B|}$.
- Some simple facts are that $n^{\alpha} = 2^{\alpha}$ whenever n is finite and α is infinite (this comes down to the fact that we can represent any element of [n] as a finite sequence of bits)
- and $\alpha^{n} = \alpha$ under the same conditions (follows by induction on n from $\alpha \cdot \alpha = \alpha$).

a combinatorial proof

- $x^a x^b = x^{a+b}$, for any cardinal numbers x, a, and b.

- Let x = |X| and let a = |A| and b = |B| where A and B are disjoint (we can always use the tagging trick that we used for inclusion-exclusion to make A and B be disjoint). Then

- $$x^a x^b = \left|X^A \times X^B\right| \text{ and } x^{a+b} = \left|X^{A \cup B}\right|$$

- bijection $f : X^{A \cup B} \to X^A \times X^B$

- The input to f is a function $g : A \cup B \to X$; the output is a pair of functions ($g_A : A \to X$, $g_B : B \to X$).

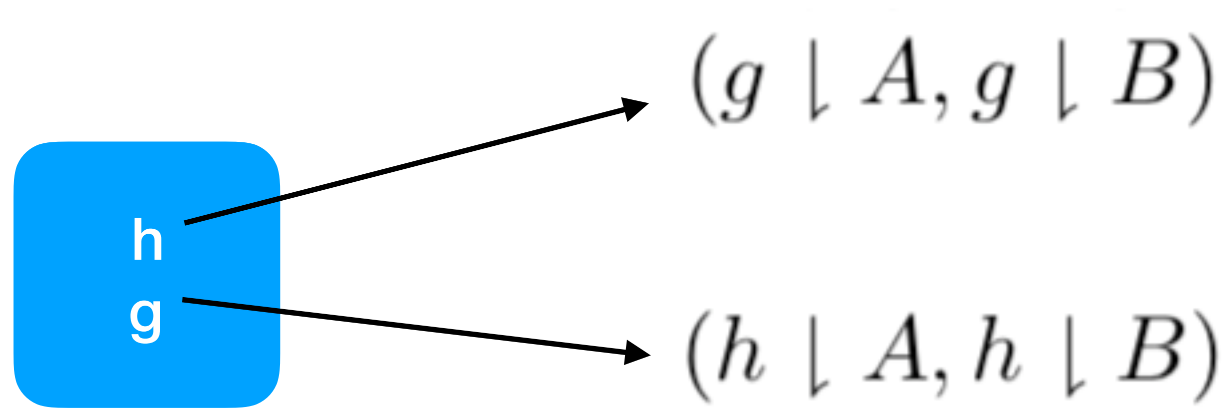
- We define g_A by $g_A(x) = g(x)$ for all x in A (this makes g_A the **restriction** of g to A, usually written as

- $g \downharpoonleft A$ or $g|A$); similarly $g_B = g \downharpoonleft B$

- This is easily seen to be a bijection

if $g = h$, then $f(g) = (g \upharpoonright A, g \upharpoonright B) = f(h) = (h \upharpoonright A, h \upharpoonright B)$

and if $g \neq h$ there is some x for which $g(x) \neq h(x)$, implying $g \upharpoonright A \neq h \upharpoonright A$ (if x is in A) or $g \upharpoonright B \neq h \upharpoonright B$ (if x is in B).



Counting injections

.Select k elements from n elements
.permute k selected elements on a sequence

- Counting injections from a k-element set to an n-element set corresponds to counting the number of ways P(n,k)

- we can pick an ordered subset of k of n items without replacement, also known as picking a k**-permutation**. (The k elements of the domain correspond to the k positions in the order.)

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.Select 3 elements from 5 elements
.permute 3 selected elements on a sequence

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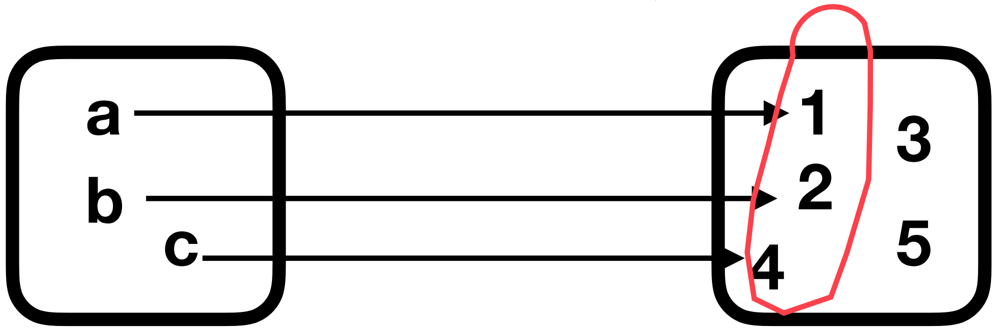
- injection

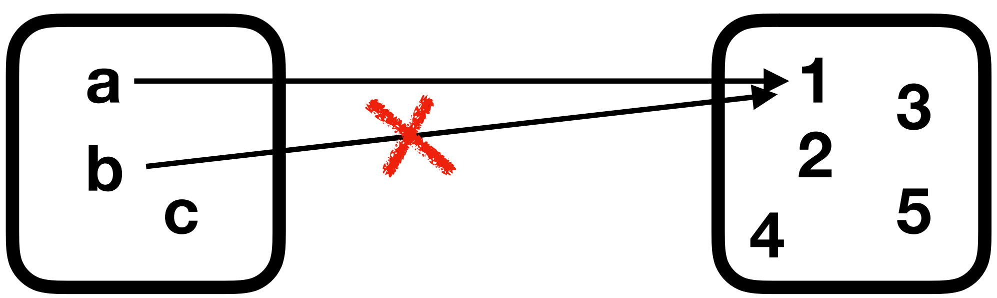


$$P(n,k) = \prod_{i=n-k+1}^{n} i = \frac{n!}{(n-k)!}$$

such k-permutations by the product rule.

Among combinatorialists, the notation $(n)_k$ (pronounced "n **lower-factorial** k") is more common than $P(n, k)$ for $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)$. As an extreme case we have $(n)_n = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-n+1) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n!$, so $n!$ counts the number of **permutations** of n.

- injection



- n^k counts the number of functions from a k-element set to an n-element set
- $(n)_k$ counts the number of injections from a k-element set to an n-element set, and
- n! counts the number of bijections between two n-element sets
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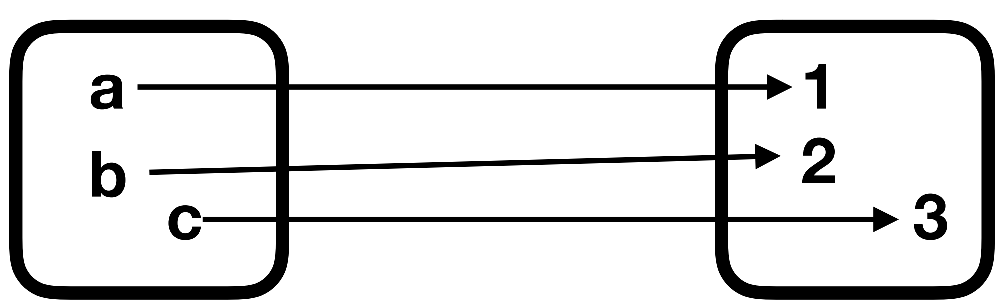
- bijection 

counting two ways

- Let $|S_k|$ denote the number of ways of choosing k elements from a set of n elements, S.
- count the number m of sequences of k elements of S with no repetitions
  - By picking a size-k subset A and then choosing one of k! ways to order the elements. This gives $m = |S_k| \cdot k!$.
  - By choosing the first element in one of n ways, the second in one of n−1, the third in one of n−2 ways, and so on until the k-th element, which can be chosen in one of n − k + 1 ways.
    - This gives $m=(n)_k=n\cdot(n-1)\cdot(n-2)\cdot\ldots(n-k+1)$, which can be written as $n!/(n-k)!$
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binomial coefficient

- So we have m = $|S_k| \cdot k! = n!/(n - k)!$, from which we get
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$$|S_k| = \frac{n!}{k! \cdot (n-k)!}.$$

This quantity turns out to be so useful that it has a special notation:

$$\binom{n}{k} \stackrel{\text{def}}{=} \frac{n!}{k! \cdot (n-k)!}.$$

Binomial coefficients

- The **binomial coefficient** "n choose k", written

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k! \cdot (n-k)!}, \qquad (11.2.1)$$

- counts the number of k-element subsets of an n-element set.

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Multinomial coefficients

- let the multinomial coeffi-cient
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$$\binom{n}{n_1\ n_2\ \dots\ n_k}$$

- be the number of different ways to distribute n items among k bins where the i-th bin gets exactly n_i of the items and we don't care what order the items appear in each bin. (Obviously this only makes sense if $n_1+n_2+\cdots+n_k=n$.)

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Two ways

- Here are two ways to count the number of permutations of the n-element set:
- 1. Pick the first element, then the second, etc., to get n! permutations.
- 2. Generate a permutation in three steps:
  - (a) Pick a partition of the n elements into blocks of size n_1, n_2, . . . n_k.
  - (b) Order the elements of each block.
  - (c) Paste the blocks together into a single ordered list.

There are

- $$\binom{n}{n_1\ n_2\ \ldots\ n_k}$$

  ways to pick the partition and

  $$n_1! \cdot n_2! \cdots n_k!$$

  ways to order the elements of all the groups, so we have

  $$n! = \binom{n}{n_1\ n_2\ \ldots\ n_k} \cdot n_1! \cdot n_2! \cdots n_k!,$$

  which we can solve to get

  $$\binom{n}{n_1\ n_2\ \ldots\ n_k} = \frac{n!}{n_1! \cdot n_2! \cdots n_k!}.$$

- a rule of the form x is in S if either P(x) or Q(x) is true
  - use the sum rule (if P and Q are mutually exclusive) or inclusion- exclusion
  - e.g. x is a tree of depth at most k if it is either (a) a single leaf node (provided k > 0) or (b) a root node with two subtrees of depth at most k–1
  - $T(k) = 1 + T(k-1)^2$ with $T(0) = 0$.[4]
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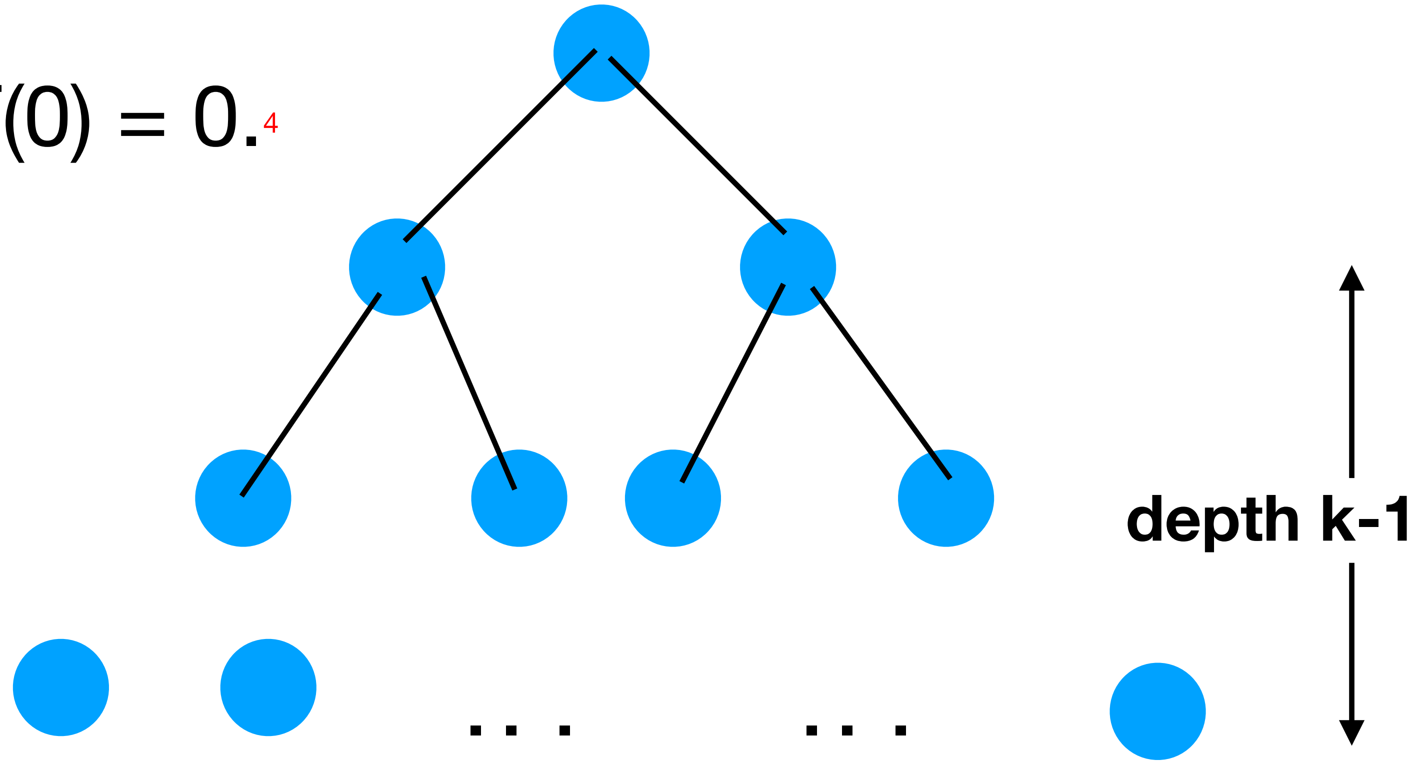

- For objects made out of many small components or resulting from many small decisions, try to reduce the description of the object to something previously known

- (a) a word of length k of letters from an alphabet of size n allowing repetition (there are n^k of them, by the product rule);

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- (b) a word of length k not allowing repetition (there are $(n)_k$ of them—or n! if n = k);
- (c) a subset of k distinct things from a set of size n, where we don't care about the order (there are $\binom{n}{k}$ of them)
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- The number of games of Tic-Tac-Toe assuming both players keep playing until the board is filled

- each such game can be specified by listing which of the 9 squares are filled in order, giving 9! = 362880 distinct games.

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- only consider games that end when one player wins

- probably the easiest way to count such games is to send a computer off to generate all of them. This gives 255168 possible games and 958 distinct final positions

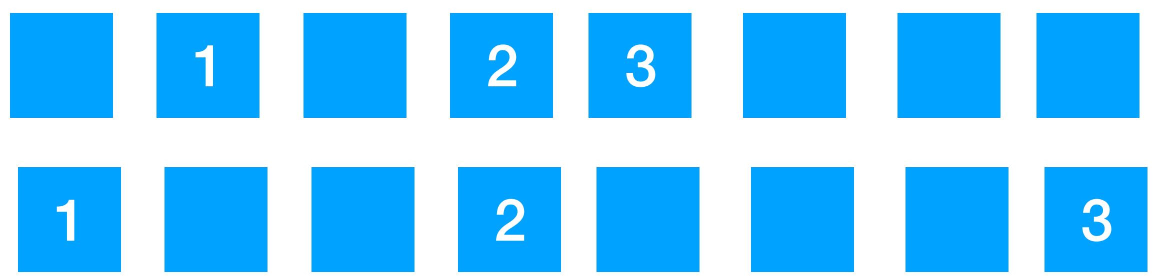
- How to count this by a program?

- https://archive.ics.uci.edu/ml/datasets/Tic-Tac-Toe+Endgame

- suppose you win n identical cars on a game show and want to divide them among your k greedy relatives

- it’s OK if some people don’t get a car at all

- putting n cars and k – 1 dividers in a line



5 cars 4 dividers

- Assume that each car—and each divider—takes one parking space. Then you have n + k – 1 parking spaces with k – 1 dividers in them (and cars in the rest). There are exactly $\binom{n+k-1}{k-1}$ ways to do this.

>> factorial(8)/(factorial(5)*factorial(3))

ans =

56

Divide 5 cars to 4 groups, then assign these four groups to 4 different relatives

- (5,0,0,0) - 4
- (4,1,0,0) - 4*3
- (3,2,0,0) - 4*3
- (3,1,1,0) - 4*3
- (2,2,1,0) - 4*3
- (2,1,1,1) - 4
- 12*4+8 = 56

- suppose you win n identical cars on a game show and want to divide them among your k greedy relatives

- Then you can just hand out one car to each relative to start with, leaving n – k cars to divide as in the previous case. There are $\binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1}$ ways to do this.

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factorial(4)/(factorial(1)*factorial(3))

ans =

4
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Divide 1 car to 4 relatives

- Divide one card to four groups, then assign these groups to 4 relatives
- (1,0,0,0) - 4
-

binomial theorem of Isaac Newton

Theorem 11.2.1 (Binomial theorem). *For any $n \in \mathbb{R}$,*

$$(x+y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k y^{n-k}, \qquad (11.2.2)$$

provided the sum converges.

A sufficient condition for the sum converging is $|x/y| < 1$. For the general version of the theorem, $\binom{n}{k}$ is defined as $(n)_k / k!$, which works even if n is not a non-negative integer. The usual proof requires calculus.

n is a non-negative integer

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

- The connection between (11.2.3) and counting subsets is straightforward: expanding $(x + y)^n$ using the distributive law gives 2^n terms, each of which is a unique sequence of n x's and y's.

- If we think of the x's in each term as labeling a subset of the n positions in the term, the terms that get added together to get $x^k y^{n-k}$ correspond one-to-one to subsets of size k.

- So there are $\binom{n}{k}$ such terms, accounting for the coefficient on the right-hand side.

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Recursive definition

Base cases:

- If $k = 0$, then there is exactly one zero-element set of our n-elem set—it's the empty set—and we have $\binom{n}{0} = 1$.
- If $k > n$, then there are no k-element subsets, and we have $\forall k >$ $\binom{n}{k} = 0$.

Recursive step: We'll use **Pascal's identity**, which says that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

- On the left-hand side, we are counting all the k-element subsets of an n-element set S.

- On the right hand side, we are counting two different collections of sets: the (k – 1)-element and k-element subsets of an (n – 1)- element set. The trick is to recognize that we get an (n – 1)-element set S' from our original set by removing one of the elements x.

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- If the subset doesn’t contain x, it doesn’t change. So there is a one- to-one correspondence (the identity function) between k-subsets of S that don’t contain x and k-subsets of S'. This bijection accounts for the first term on the right-hand side.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

- If the subset does contain x, then we get a (k – 1)-element subset of S' when we remove it. Since we can go back the other way by reinserting x, we get a bijection between k-subsets of S that contain x and (k – 1)-subsets of S'. This bijection accounts for the second term on the right-hand side.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

re's the proof of Pascal's identity:

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} x^k &= (1+x)^n \\
&= (1+x)(1+x)^{n-1} \\
&= (1+x)^{n-1} + x(1+x)^{n-1} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^k + x \sum_{k=0}^{n-1} \binom{n-1}{k} x^k \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^k + \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^k + \sum_{k=1}^{n} \binom{n-1}{k-1} x^k \\
&= \sum_{k=0}^{n} \binom{n-1}{k} x^k + \sum_{k=0}^{n} \binom{n-1}{k-1} x^k \\
&= \sum_{k=0}^{n} \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right) x^k .
\end{aligned}$$

Vandermonde’s identity

Vandermonde’s identity says that, provided r does not exceed m or r

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}.$$

- To pick r elements of an m + n element set, we have to pick some of them from the first m elements and some from the second n elements.

- we choose k elements from the last n

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$$\left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{i=0}^{m} b_i x^i\right) = \sum_{i=0}^{m+n}\left(\sum_{j=0}^{i} a_j b_{i-j}\right) x^i.$$

So now consider

$$\begin{aligned}
\sum_{r=0}^{m+n}\binom{m+n}{r}x^r &= (1+x)^{m+n} \\
&= (1+x)^n(1+x)^m \\
&= \left(\sum_{i=0}^{n}\binom{n}{i}x^i\right)\left(\sum_{j=0}^{m}\binom{m}{j}x^j\right) \\
&= \sum_{r=0}^{m+n}\left(\sum_{k=0}^{r}\binom{n}{k}\binom{m}{r-k}\right)x^r.
\end{aligned}$$