Lecture 3

function

- A **function symbol** looks like a predicate but instead of computing a truth value it returns an object.

- Function symbols may take zero or more arguments. The special case of a function symbol with zero arguments is called a **constant**.

- These are represented using the constant 0 and the **successor function** S, so that we can count 0, S0, SS0, SSS0, and so on.

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equality

- The **equality** predicate =, written x = y, is typically included as a standard part of predicate logic.
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- The interpretation of $x = y$ is that x and y are the same element of the domain.

- Equality satisfies the **reflexivity axiom** $\forall x : x = x$ and the **substitution axiom schema**: $\forall x \forall y : (x = y \rightarrow (Px \leftrightarrow Py))$, where P is any predicate. This immediately gives a **substitution rule** that says $x = y, P(x) \vdash P(y)$.

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- **substitution axiom schema**: $\forall x \forall y : (x = y \rightarrow (Px \leftrightarrow Py))$

- $\forall x \forall y : (x = y \rightarrow y = x)$ from the above axioms (this property is known as **symmetry**).

- Apply substitution to the predicate $Pz \equiv z=x$ to get $\forall x \forall y:(x=y \rightarrow (x=x \leftrightarrow y=x))$.

- Use reflexivity to rewrite this as $\forall x \forall y : (x = y \rightarrow (1 \leftrightarrow y = x))$, which simplifies to

  $\forall x \forall y : (x = y \rightarrow y = x)$.

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Uniqueness

- The abbreviation $\exists!x\, P(x)$ says "there exists a *unique* x such that P(x)." This is short for
  $\exists x(P(x) \wedge (\forall y : P(y) \rightarrow x = y))$,
- There is an x for which P (x) is true, and any y for which P(y) is true is equal to x."

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- There are several equivalent ways to expand $\exists!x\, P(x)$.

- Applying contraposition to $P(y) \rightarrow x = y$ gives
  $\exists!xP(x) \equiv \exists x(P(x) \wedge (\forall y : x \neq y \rightarrow \neg P(y)))$,
  which says that any y that is not x doesn't satisfy P.

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- De Morgan’s laws to turn this into $\exists! x\, P(x) \equiv \exists x(P(x) \wedge (\neg\exists y : x \neq y \wedge P(y)))$.

- This says that there is an x with P(x), but there is no $y \neq x$ with P(y). All of these are just different ways of saying that x is the only object that satisfies P.

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model

- Consider the axiom $\neg\exists x$. This axiom has exactly one model (it's empty).

- Now consider the axiom $\exists! x$, which we can expand out to $\exists x \forall y\ y = x$.
  This axiom also has exactly one model (with one element).

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Models

- A structure is a **model** of a particular **theory** (set of statements), if each statement in the theory is true in the model.

- We can enforce exactly k elements with one rather long axiom, e.g. for k=3 do

$$\exists x_1 \exists x_2 \exists x_3 \forall y : y = x_1 \vee y = x_2 \vee y = x_3 \wedge x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_3 \neq x_1$$

- In the absence of any special symbols, a structure of 3 undifferentiated elements is the unique model of this axiom.

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- Suppose we add a predicate P and consider the axiom $\exists x Px$.

- Let P be true of at least one of its elements.

- If we take a model with two elements a and b, with Pa and $\neg$Pb, we see that $\exists x Px$ is not enough to prove $\forall x Px$, since $\exists x P x$ is true in the model but $\forall x P x$ isn't.

- Conversely, an empty model satisfies $\forall x Px \equiv \neg\exists x \neg Px$ but not $\exists x Px$.

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- A practical example: The family tree of the kings of France is a model of the theory containing the two axioms.

  - $\forall x \forall y \forall z Parent(x, y) \wedge Parent(y, z) \rightarrow GrandParent(x, z)$

  - $\forall x \forall y Parent(x, y) \rightarrow \neg Parent(y, x)$.

- But this set of axioms could use some work, since it still allows for the possibility that there are some x and y for which Parent(x, y) and GrandParent(y, x) are both true.

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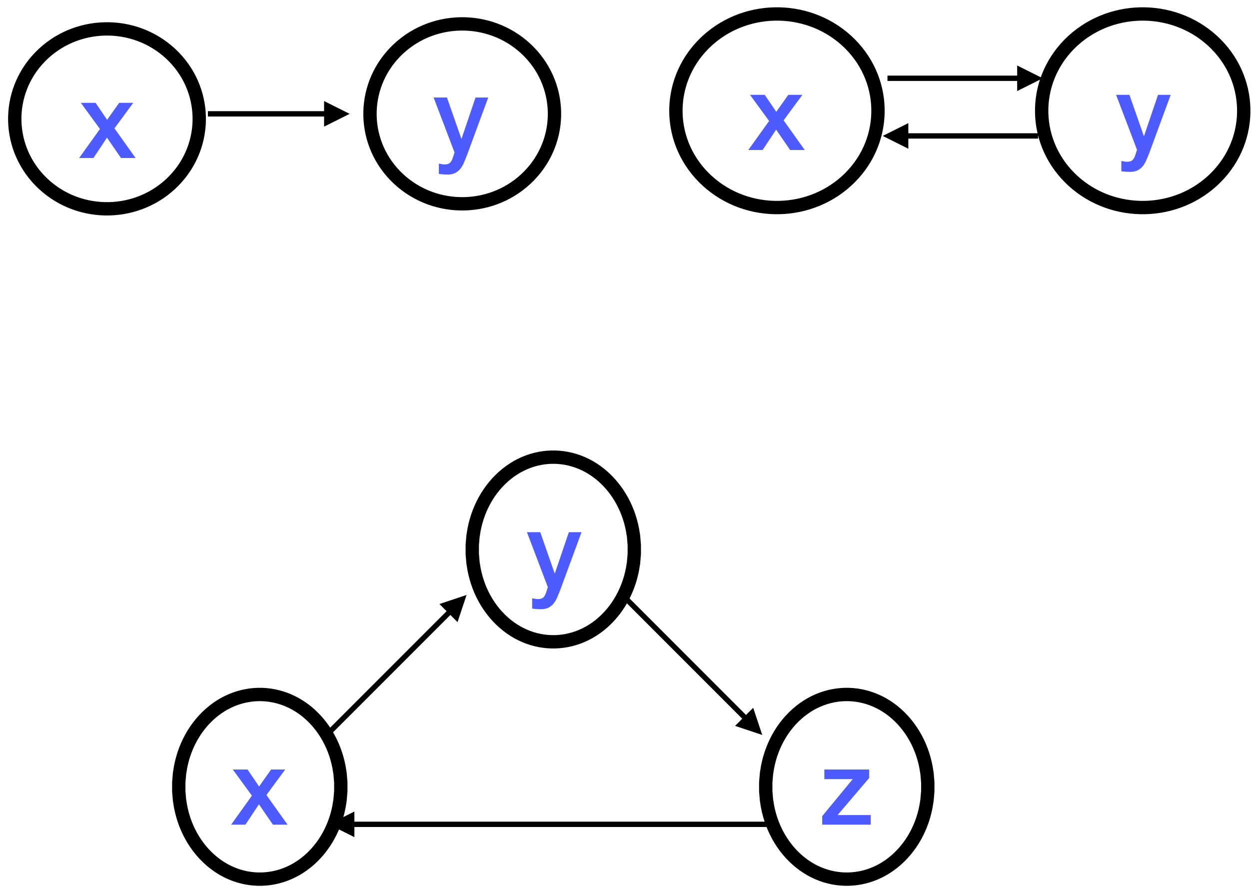
parent function



Proofs

- A proof is a way to derive statements from other statements.

- It starts with **axioms** (statements that are assumed in the current context always to be true), **theorems** or **lemmas** (statements that were proved already; the difference between a theorem and a lemma is whether it is intended as a final result or an intermediate tool), and **premises** P (assumptions we are making for the purpose of seeing what consequences they have), and uses **inference rules** to derive Q.

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- The axioms, theorems, and premises are in a sense the starting position of a game whose rules are given by the inference rules. The goal of the game is to apply the inference rules until Q pops out.

- We refer to anything that isn’t proved in the proof itself (i.e., an axiom, theorem, lemma, or premise) as a **hypothesis**; the result Q is the **conclusion**.

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- When a proof exists of Q from some premises P_1, P_2, . . . , we say that Q is **deducible** or **provable** from P_1, P_2, . . . , which is written as
$P_1, P_2, \ldots \vdash Q$.

- If we can prove Q directly from our inference rules without making any assumptions, we may write
$\vdash Q$

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- The **turnstile** symbol $\vdash$ has the specific meaning that we can derive the conclusion Q by applying inference rules to the premises.

- This is not quite the same thing as saying $P \to Q$.

- If our inference rules are particularly weak, it may be that $P \to Q$ is true but we can't prove Q starting with P.

- Conversely, if our inference rules are too strong (maybe they can prove anything, even things that aren't true) we might have $P \vdash Q$ but $P \to Q$ is false.

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- For propositions, most of the time we will use inference rules that are just right, meaning that $P \vdash Q$ implies that $P \rightarrow Q$ is a tautology, (**soundness**) and $P \rightarrow Q$ being a tautology implies that $P \vdash Q$ (**completeness**).

- Here the distinction between $\vdash$ and $\rightarrow$ is whether we want to talk about the existence of a proof (the first case) or about the logical relation between two statements (the second).

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Inference Rules

- Inference rules let us construct **valid** arguments, which have the useful property that if their premises are true, their conclusions are also true.
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- The main source of inference rules is tautologies of the form $P_1 \wedge P_2 \ldots \rightarrow Q$; given such a tautology, there is a corresponding inference rule that allows us to assert Q once we have $P_1, P_2, \ldots$.

- Given an inference rule of this form and a goal Q, we can then look for ways to show $P_1, P_2, \ldots$ all hold, either because each P_i is an axiom/theorem/premise or because we can prove it from other axioms, theorems, or premises.

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- The most important inference rule is **modus ponens**, ’ based on the tautology $(p \wedge (p \to q)) \to q$; this lets us, for example, write the following famous argument:[9]

- 1. If it doesn’t fit, you must acquit. [Axiom]
  2. It doesn’t fit. [Premise]
  3. You must acquit. [Modus ponens applied to 1+2]

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- The "addition" rule below is just the result of applying modus ponens to p and the tautology $p \rightarrow (p \vee q)$

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- Premises are listed on the left-hand side separated by commas, and the conclusion is placed on the right. We can then write

$p \vdash p \lor q.$	Addition
$p \land q \vdash p.$	Simplification
$p, q \vdash p \land q.$	Conjunction
$p, p \to q \vdash q.$	Modus ponens
$\neg q, p \to q \vdash \neg p.$	Modus tollens
$p \to q, q \to r \vdash p \to r.$	Hypothetical syllogism
$p \lor q, \neg p \vdash q.$	Disjunctive syllogism
$p \lor q, \neg p \lor r \vdash q \lor r.$	Resolution

- Modus ponens “the method of affirming” (and its reversed cousin modus tollens “the method of denying”)
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- It does not necessarily follow that the conclusion is true; it could be that one or more of the hypotheses is false:

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1. If you give a mouse a cookie, he's going to ask for a glass of milk. [Axiom]

2. If he asks for a glass of milk, he will want a straw. [Axiom]

3. You gave a mouse a cookie. [Premise]

4. He asks for a glass of milk. [Modus ponens applied to 1 and 3.]

5. He will want a straw. [Modus ponens applied to 2 and 4.]

Will the mouse want a straw? No: Mice can't ask for glasses of milk, so Axiom 1 is false.

- Recall that P ⊢ Q means there is a proof of Q by applying inference rules to P , while P → Q says that Q holds whenever P does.

- These are not the same thing: provability (⊢) is outside the theory (it’s a statement about whether a proof exists or not) while implication (→) is inside (it’s a logical connective for making compound propositions). But most of the time they mean almost the same thing.

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Theorem 2.4.1 (Deduction Theorem). *If there is a proof of Q from premises $\Gamma, P_1, P_2, \ldots, P_n$, then there is a proof of $P_1 \wedge P_2 \wedge \ldots \wedge P_n \to Q$ from Γ alone.*

- $$\Gamma, P_1, P_2, \ldots, P_n \vdash Q$$

$$\Gamma \vdash (P_1 \wedge P_2 \wedge \ldots \wedge P_n) \to Q.$$

- This style of inference rule, where we explicitly track what assumptions go into a particular result, is known as **natural deduction**.

- The natural deduction approach was invented by Gentzen [Gen35a, Gen35b] as a way to make inference rules more closely match actual mathematical proof-writing practice than the modus-ponens-only approach that modern logicians had been using up to that point.[10]

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Natural deduction

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q} \qquad (\to I)$$

- if we can prove Q using assumptions Γ and P, then we can prove P → Q using just Γ
- introducing implication
- Note that the horizontal line acts like a higher-order version of ⊢; it lets us combine one or more proofs into a new, bigger proof.
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- eliminating implication that is essentially just modus ponens:

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$$\frac{\Gamma \vdash P \to Q \quad \Gamma \vdash P}{\Gamma \vdash Q} \quad (\to E)$$

substitution rule

$$x = y, P(x) \vdash P(y).$$

- an axiom schema:
  ∀x : ∀y : ((x = y ∧ P (x)) → P (y)).
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Natural deduction: introduction and elimination rules

$$\frac{\Gamma \vdash P}{\Gamma \vdash \neg\neg P} \quad (\neg I)$$

$$\frac{\Gamma \vdash \neg\neg P}{\Gamma \vdash P} \quad (\neg E)$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q} \quad (\wedge I)$$

$$\frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash P} \quad (\wedge E_1)$$

$$\frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash Q} \quad (\wedge E_2)$$

Natural deduction: introduction and elimination rules

$$\frac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q} \qquad (\vee I_2)$$

$$\frac{\Gamma \vdash P \vee Q \quad \Gamma \vdash \neg Q}{\Gamma \vdash P} \qquad (\vee E_1)$$

$$\frac{\Gamma \vdash P \vee Q \quad \Gamma \vdash \neg P}{\Gamma \vdash Q} \qquad (\vee E_2)$$

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q} \qquad (\to I)$$

$$\frac{\Gamma \vdash P \to Q \quad \Gamma \vdash P}{\Gamma \vdash Q} \qquad (\to E_1)$$

$$\frac{\Gamma \vdash P \to Q \quad \Gamma \vdash \neg Q}{\Gamma \vdash \neg P} \qquad (\to E_2)$$

Inference rules for quantified statements

- **Universal generalization** If y is a variable that does not appear in Γ, then

$$\frac{\Gamma \vdash P(y)}{\Gamma \vdash \forall x : P(x)}$$

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- This says that if we can prove that some property holds for a “generic” y, without using any particular properties of y, then in fact the property holds for all possible x.

- In a written proof, this will usually be signaled by starting with some- thing like “Let y be an arbitrary [member of some universe]”. For example: Suppose we want to show that there is no biggest natural number,i.e. that

$\forall n \in N : \exists n' \in N : n' > n$. Proof: Let n be any element of N. Let $n' = n+1$. Then $n' > n$.

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- **Universal instantiation** In the other direction, we have $\forall x : Q(x) \vdash Q(c)$.

- Here we go from a general statement about all possible values x to a statement about a particular value. Typical use: Given that all humans are mortal, it follows that Spocrates is mortal.

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Existential generalization

This is essentially the reverse of universal instantiation: it says that, if c is some particular object, we get

$$Q(c) \vdash \exists x : Q(x).$$

The idea is that to show that Q(x) holds for at least one x, we can point to c as a specific example of an object for which Q holds.

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- **Existential instantiation**

- $\exists x : Q(x) \vdash Q(c)$ for some c,

where c is a new name that hasn't previously been used (this is similar to the requirement for universal generalization, except now the new name is on the right-hand side).

- The idea here is that we are going to give a name to some c that satisfies Q(c), and we know that we can get away this because $\exists x : Q(x)$ says that some such thing exists.

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Proof techniques

$$\frac{\Gamma \vdash Pc}{\Gamma \vdash \forall x : Px} \qquad (\forall I)$$

$$\frac{\Gamma \vdash \forall x : Px}{\Gamma \vdash Pc} \qquad (\forall E)$$

$$\frac{\Gamma \vdash Pc}{\Gamma \vdash \exists x : Px} \qquad (\exists I)$$

$$\frac{\Gamma \vdash \exists x : Px}{\Gamma \vdash Pc} \qquad (\exists E)$$

Table 2.5: Natural deduction: introduction and elimination rules for quantifiers. For $\forall I$ and $\exists E$, c is a new symbol that does not appear in P or Γ.

- Table 2.6 gives techniques for trying to prove A → B for particular statements A and B. The techniques are mostly classified by the structure of B. Before applying each technique, it may help to expand any definitions that appear in A or B.
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Strategy	When	Assume	Conclude	What to do/why it works
Direct proof	Try it first	A	B	Apply inference rules to work forward from A and backward from B; when you meet in the middle, pretend that you were working forward from A all along.
Contraposition	$B = \neg Q$	$\neg B$	$\neg A$	Apply any other technique to show $\neg B \to \neg A$ and then apply the contraposition rule. Sometimes called an *indirect proof* although the term *indirect proof* is often used instead for proofs by contradiction (see below).

Construction	$B = \exists x P(x)$	A	$P(c)$ for some specific object c.	Pick a likely-looking c and prove that $P(c)$ holds.
Counterexample	$B = \neg\forall x P(x)$	A	$\neg P(c)$ for some specific object c.	Pick a likely-looking c and show that $\neg P(c)$ holds. This is identical to a proof by construction, except that we are proving $\exists x \neg P(x)$, which is equivalent to $\neg\forall x P(x)$.

Choose	$B = \forall x(P(x) \to Q(x))$	A, $P(c)$, where c is chosen arbitrarily.	$Q(c)$	Choose some c and assume A and $P(c)$. Prove $Q(c)$. Note: c is a placeholder here. If $P(c)$ is "c is even" you can write "Let c be even" but you can't write "Let $c = 12$", since in the latter case you are assuming extra facts about c.

∃

Instantiation	$A = \forall x P(x)$	A	B	Pick some particular c and prove that $P(c)$ →B. Here you *can* get away with saying "Let $c = 12$." (If $c = 12$ makes B true).
Elimination	$B = C \vee D$	$A \wedge \neg C$	D	The reason this works is that $A \wedge \neg C \rightarrow D$ is equivalent to $\neg(A \wedge \neg C) \rightarrow D \equiv \neg A \vee C \vee D \equiv A \rightarrow (C \vee D)$. Of course, it works equally well if you start with $A \wedge \neg D$ and prove C.

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Case analysis	$A = C \vee D$	C, D	B	Here you write two separate proofs: one that assumes C and proves B, and one that assumes D and proves B. A special case is when $D = \neg C$. You can also consider more cases, as long as A implies at least one of the cases holds.
Induction	$B = \forall x \in \mathbb{N} P(x)$	A	$P(0)$ and $\forall x \in \mathbb{N} : (P(x) \rightarrow P(x+1))$.	If $P(0)$ holds, and $P(x)$ implies $P(x+1)$ for all x, then for any specific natural number n we can consider constructing a sequence of proofs $P(0) \rightarrow P(1) \rightarrow P(2) \rightarrow \ldots \rightarrow P(n)$. (This is actually a defining property of the natural numbers.)

- Ex means that x is even

$$A_1 : \forall x : Ex \leftrightarrow (x = 0 \vee (\exists y : Ey \wedge x = SSy))$$
$$A_2 : \forall x : 0 \neq Sx.$$
$$A_3 : \forall x \forall y : Sx = Sy \to x = y.$$

Here A_1 is the definition of Ex and A_2 and A_3 are general axioms about S that we are throwing in because we will need them in some of our proofs.

Theorem 2.6.1. *All of the following statements are true:*

1. $E0$.
2. $\neg E(S0)$.
3. $E(SS0)$.
4. $\neg E(SSS0)$.
5. $E(SSSS0)$.

1. $E0$.

2. $\neg E(S0)$.

3. $E(SS0)$.

Proof. 1. Axiom A_1 says that x is even if it is 0.

2. Suppose $E(S0)$ holds. Then either $S0 = 0$ or $S0 = SSy$ for some y such that Ey holds. The first case contradicts A_2; in the second case, applying A_3 gives that $S0 = SSy$ implies $0 = Sy$, which again contradicts A_2. So in either case we arrive at a contradiction, and our original assumption that $E(S0)$ is true does not hold.

   (This is an example of an indirect proof.)

3. From A_1 we have that $E(SS0)$ holds if there exists some y such that Ey and $SS0 = SSy$. Let $y = 0$.

4. $\neg E(SSS0)$.

5. $E(SSSS0)$.

4. We have previously established $\neg E(S0)$. We also know that $SSS0 \neq 0$, so $E(SSS0)$ is true if and only if $SSS0 = SSy$ for some y with Ey. Applying A_2 twice gives $SSS0 = SSy$ iff $S0 = y$. But we already showed $\neg E(S0)$, so $\neg E(SSS0)$.

5. Since $E(SS0)$ and $SSSS0 = SS0$, $E(SSSS0)$.

□

- Let's try this for the proof of ¬E(S0).

- We are trying to establish that $A_1, A_2, A_3 \vdash \neg E(S0)$. Abbreviating A_1, A_2, A_3 as Γ, the strategy is to show that

$\Gamma \vdash E(S0) \rightarrow Q$ for some Q with $\Gamma \vdash \neg Q$;

we can then apply the $\rightarrow E_2$ rule (aka modus tollens) to get

$\Gamma \vdash \neg E(S0)$.

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$$\frac{\Gamma \vdash P \rightarrow Q \quad \Gamma \vdash \neg Q}{\Gamma \vdash \neg P} \quad (\rightarrow E_2)$$

1. $\Gamma \vdash E(S0) \leftrightarrow (S0 = 0 \vee \exists y : (Ey \wedge S0 = SSy))$. ($\forall E$ applied to A_1.)

2. $\Gamma \vdash E(S0) \rightarrow (S0 = 0 \vee \exists y : (Ey \wedge S0 = SSy))$. (Expand $\leftrightarrow$ and use one of the $\wedge$ elimination rules.)

3. $\Gamma, E(S0) \vdash S0 = 0 \vee \exists y : (Ey \wedge S0 = SSy)$. ($\rightarrow E$).

$$A_1 : \forall x : Ex \leftrightarrow (x = 0 \vee (\exists y : Ey \wedge x = SSy))$$

$$\frac{\Gamma \vdash \forall x : Px}{\Gamma \vdash Pc} \qquad (\forall E)$$

$$\frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash P} \qquad (\wedge E_1)$$

$$\frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash Q} \qquad (\wedge E_2)$$

$$\frac{\Gamma \vdash P \rightarrow Q \quad \Gamma \vdash P}{\Gamma \vdash Q} \qquad (\rightarrow E)$$

3. $\Gamma, E(S0) \vdash S0 = 0 \lor \exists y : (Ey \land S0 = SSy)$. $(\to E)$.

4. $\Gamma, E(S0) \vdash \neg(S0 = 0)$. (Apply $\forall E$ to A_2.)

5. $\Gamma, E(S0) \vdash \exists y : (Ey \land S0 = SSy)$. (Combine last two steps using $\lor E_1$.)

6. $\Gamma, E(S0) \vdash Ez \land S0 = SSz$. (This is $\exists E$. In the condensed proof we didn't rename y, but calling it z here makes it a little more obvious that we are fixing some particular constant.)

$$A_2 : \forall x : 0 \neq Sx.$$

$$\frac{\Gamma \vdash \forall x : Px}{\Gamma \vdash Pc} \qquad (\forall E)$$

$$\frac{\Gamma \vdash P \lor Q \quad \Gamma \vdash \neg P}{\Gamma \vdash Q} \qquad (\lor E_2)$$

$$\frac{\Gamma \vdash \exists x : Px}{\Gamma \vdash Pc} \qquad (\exists E)$$

8. $\Gamma, E(S0) \vdash S0 = SSz \leftrightarrow 0 = Sz$. (Apply $\forall E$ to A_3).

9. $\Gamma, E(S0) \vdash S0 = SSz \to 0 = Sz$. (Another expansion plus $\wedge E$).

10. $\Gamma, E(S0) \vdash 0 = Sz$. (Apply $\to E_1$ to $S0 = SSz$ and $S0 = SSz \to 0 = Sz$.)

$$A_3 : \forall x \forall y : Sx = Sy \to x = y.$$

$$\frac{\Gamma \vdash \forall x : Px}{\Gamma \vdash Pc} \qquad (\forall E)$$

$$\frac{\Gamma \vdash P \to Q \quad \Gamma \vdash P}{\Gamma \vdash Q} \qquad (\to E_1)$$

$\leftrightarrow$

10. $\Gamma, E(S0) \vdash 0 = Sz$. (Apply $\to E_1$ to $S0 = SSz$ and $S0 = SSz \to 0 = Sz$.)

11. $\Gamma \vdash E(S0) \to 0 = Sz$. ($\to I$.)

12. $\Gamma \vdash \neg(0 = Sz)$. ($\forall E$ and A_2.)

13. $\Gamma \vdash \neg E(S0)$. ($\to E_2$.)

$$A_2 : \forall x : 0 \neq Sx.$$

$$\frac{\Gamma \vdash \forall x : Px}{\Gamma \vdash Pc} \qquad (\forall E)$$

$$\frac{\Gamma \vdash P \to Q \quad \Gamma \vdash \neg Q}{\Gamma \vdash \neg P} \qquad (\to E_2)$$

Theorem 2.6.2. *For all x, if x is even, $SSSSx$ is even.*

Proof. Let x be even. Then SSx is even (Axiom A_1), and so $SS(SSx) = SSSSx$ is also even. ∎

Written out using natural-deduction inference rules (with some of the more boring steps omitted), the proof would look like this:

1. $\Gamma, Ex \vdash (\exists y : Ey \wedge SSx = SSy) \to E(SSx)$. (Axiom A_1, $\forall E$, $\vee E_1$.)

2. $\Gamma, Ex \vdash Ex$.

3. $\Gamma, Ex \vdash SSx = SSx$. (Reflexivity of $=$.)

4. $\Gamma, Ex \vdash Ex \wedge SSx = SSx$. ($\wedge I$ applied to previous two steps.)

5. $\Gamma, Ex \vdash \exists y : Ey \wedge SSy = SSx$. (Let $y = x$.)

6. $\Gamma, Ex \vdash E(SSx)$. (Modus ponens!)

7. $\Gamma, Ex \vdash E(SSSSx)$. (Do it all again to show $E(SSx) \to E(SSSSx)$. This is the boring part we promised to omit.)

8. $\Gamma \vdash Ex \to E(SSSSx)$. ($\to I$.)

9. $\Gamma \vdash \forall x : Ex \to E(SSSSx)$. ($\forall I$).

1. $\Gamma, Ex \vdash (\exists y : Ey \wedge SSx = SSy) \to E(SSx)$. (Axiom A_1, $\forall E$, $\vee E_1$.)
2. $\Gamma, Ex \vdash Ex$.
3. $\Gamma, Ex \vdash SSx = SSx$. (Reflexivity of =.)

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SSx

$$A_1 : \forall x : Ex \leftrightarrow (x = 0 \vee (\exists y : Ey \wedge x = SSy))$$

$$\frac{\Gamma \vdash \forall x : Px}{\Gamma \vdash Pc} \quad (\forall E)$$

$$\frac{\Gamma \vdash P \vee Q \quad \Gamma \vdash \neg Q}{\Gamma \vdash P} \quad (\vee E_1)$$

1. $\Gamma, Ex \vdash (\exists y : Ey \wedge SSx = SSy) \to E(SSx)$. (Axiom A_1, $\forall E$, $\vee E_1$.)

2. $\Gamma, Ex \vdash Ex$.

3. $\Gamma, Ex \vdash SSx = SSx$. (Reflexivity of =.)

4. $\Gamma, Ex \vdash Ex \wedge SSx = SSx$. ($\wedge I$ applied to previous two steps.)

5. $\Gamma, Ex \vdash \exists y : Ey \wedge SSy = SSx$. (Let $y = x$.)

6. $\Gamma, Ex \vdash E(SSx)$. (Modus ponens!)

7. $\Gamma, Ex \vdash E(SSSSx)$. (Do it all again to show $E(SSx) \to E(SSSSx)$. This is the boring part we promised to omit.)

8. $\Gamma \vdash Ex \to E(SSSSx)$. ($\to I$.)

9. $\Gamma \vdash \forall x : Ex \to E(SSSSx)$. ($\forall I$).